



ENGINEERING PHYSICS AND MATHEMATICS

Effects of variable viscosity and thermal conductivity on MHD flow and heat transfer of a dusty fluid



S. Manjunatha ^{a,*}, B.J. Gireesha ^b

^a Department of Mathematics, Acharya Institute of Technology, Soladevanhalli, Bangalore 560107, Karnataka, India

^b Department of Studies and Research in Mathematics, Kuvempu University, Shankaraghatta 577 451, Shimoga, Karnataka, India

Received 16 July 2014; revised 12 December 2014; accepted 17 January 2015

Available online 27 May 2015

KEYWORDS

Boundary layer flow;
Stretching surface;
Temperature-dependent fluid viscosity;
Dusty fluid;
Thermal-diffusion;
Numerical solution

Abstract The problem of magnetohydrodynamic flow and heat transfer of a viscous, incompressible and electrically conducting dusty fluid over an unsteady stretching sheet is analyzed numerically. The fluid viscosity and thermal conductivity are assumed to vary as an exponential function of temperature. The governing fundamental equations are approximated by a system of nonlinear ordinary differential equations using similarity transformations. The obtained similarity equations are solved numerically using RKF-45 method. Numerical computation has been carried out for horizontal velocity profiles, temperature, Nusselt number and skin friction coefficient for various values of the flow parameters that are presented for both VWT and VHF respectively. A comparison with previously published work is performed and the results are found to be in good agreement.

© 2015 Faculty of Engineering, Ain Shams University. Production and hosting by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

1. Introduction

During past few decades, considerable interest has been given to study the flow of an incompressible viscous fluid over a continuous moving solid surface which has important applications in a variety of manufacturing processes. For instance, a number of technical processes concerning polymers involve the

cooling of continuous strips extruded from a die by drawing them through a cooling bath and in the process of drawing, these strips are subsequently stretched to achieve the desired thickness. The quality of final product depends on the rate of heat transfer and therefore cooling procedure has to be controlled effectively. The MHD flow in electrically conducting fluid can control the rate of cooling and the desired quality of product can be achieved [1]. The steady flow on a moving continuous flat surface was first considered by Sakiadis [2] who has developed a numerical solution using a similarity transformation. Chiam [3] reported solutions for steady hydro-magnetic flow over a surface stretching with a power law velocity with the distance along the surface. Tsou et al. [4] studied a wide ranging analytical and experimental investigation of the flow and heat transfer characteristics of the boundary layer on a continuous moving surface. The two-dimensional flow

* Corresponding author. Tel.: +91 9945829702.

E-mail addresses: manjubhushana@gmail.com (S. Manjunatha), bigireesu@rediffmail.com (B.J. Gireesha).

Peer review under responsibility of Ain Shams University.



Production and hosting by Elsevier

caused solely by a linearly stretching sheet in an incompressible fluid which has a very simple closed form exponential solution and was established by Crane [5]. The temperature field in the flow over stretching surface subject to a uniform heat flux was studied by Grubka and Bobba [6], while Elbashebeshy et al. [7,8] considered the case of stretching surface with a variable surface heat flux. Further they have presented similarity solutions of the boundary layer equations, which describe the unsteady flow and heat transfer over a stretching sheet. Sharidan et al. [9] investigated an unsteady flow and heat transfer of a viscous and incompressible fluid over a stretching sheet.

All the abovementioned studies continued their discussions by assuming the physical properties of the ambient fluid were to be constants. However, it is known that the physical properties of fluid may change significantly with temperature, especially for fluid viscosity (see [10–13]). The increase of temperature leads to the increase in the transport phenomena by reducing the viscosity across the momentum boundary layer and due to which the heat transfer rate at the wall is also affected. Therefore, to predict the flow and heat transfer rates it is necessary to take into account the temperature dependent viscosity of the fluid. For lubricating fluids heat generated by internal friction and the corresponding rise in the temperature affects the viscosity of the fluid and so that the fluid viscosity no longer be assumed constant. McCormack and Crane [14] have studied and gave comprehensive discussion on boundary layer flow caused by stretching of an elastic flat sheet moving in its own plane with a velocity varying linearly with distance. Salem [15] investigated variable viscosity and thermal conductivity effects on MHD flow and heat transfer in viscoelastic fluid over a stretching sheet. Chiam [16] considered the effect of variable thermal conductivity on the flow and heat transfer from a linearly stretching sheet. Pantokratoras [17] presented further results on the variable viscosity on the flow and heat transfer to a continuous moving flat plate. Mukhopadhyay and Layek [18] analyzed the effect of thermal radiation and variable fluid viscosity on free convective and heat transfer past a porous stretching surface.

The above literature describes the boundary layer flow of fluid phase only. Vajravelu and Nayfeh [19] discussed hydromagnetic flow of a dusty fluid over a stretching sheet. Recently Gireesha et al. [20,21] have studied hydromagnetic boundary layer flow and heat transfer of dusty fluid over a stretching sheet for both steady and unsteady flow.

In view of these, the present problem is concentrated on the fluid properties which depend on high temperature. Thus the main motivation of the present paper was to study the effects of variable fluid viscosity and thermal conductivity on MHD boundary layer flow and heat transfer of dusty fluid past a stretching sheet. The fluid is assumed to be viscous and incompressible. Consideration of temperature-dependent viscosity, thermal conductivity and magnetic parameter yields a highly non-linear coupled system of partial differential equations. The coupled non-linear partial differential equations governing the problem are reduced to a system of coupled highly non-linear higher-order ordinary differential equations by applying suitable similarity transformations. The system of higher order ordinary differential equations is then solved by employing Runge–Kutta–Fehlberg-45 method. Attempts have been made to investigate the effect of variable fluid viscosity, magnetic parameter, variable thermal conductivity parameter,

fluid interaction parameter, Prandtl number and Eckert number on the flow behavior and heat transfer process.

2. Mathematical formulation of the problem

Consider an unsteady boundary layer flow and heat transfer of a viscous incompressible and electrically conducting dusty fluid past a stretching surface coinciding with the plane $y = 0$ and the flow being confined to $y > 0$. The x -axis is taken in the direction of the main flow along the sheet and y -axis is normal to it. Keeping the origin fixed, two equal and opposite forces are applied along the in the x -axis which results in stretching of the sheet and hence, the flow is generated as shown in Fig. 1. The continuous sheet moves in its own plane with velocity $u = u_w = \frac{cx}{1-qt}$, and the temperature distribution varies along both the sheet and time. The fluid moves in the x -direction with a velocity (u -component) equal to the velocity of the solid surface and the velocity of the fluid in the x -direction approaches to zero asymptotically. The uniform transverse magnetic field B_0 is imposed along the positive y -axis. The magnetic Reynolds number is assumed to be small so that the induced magnetic field is neglected in comparison with the applied magnetic field. Both the fluid and dust particle clouds are supposed to be static at the beginning and the dust particles are assumed to be spherical in shape and uniform in size. The number density of these is taken to be a constant throughout the flow. Under the foregoing assumptions and invoking the usual boundary layer approximation, the governing equations for momentum and heat transfer of dusty fluid in the presence of variable fluid properties (fluid viscosity and thermal conductivity) take the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) + \frac{kN}{\rho} (u_p - u) - \frac{\sigma B_0^2}{\rho} u, \tag{2.2}$$

$$\frac{\partial u_p}{\partial t} + u_p \frac{\partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y} = \frac{k}{m} (u - u_p), \tag{2.3}$$

$$\frac{\partial v_p}{\partial t} + u_p \frac{\partial v_p}{\partial x} + v_p \frac{\partial v_p}{\partial y} = \frac{k}{m} (v - v_p), \tag{2.4}$$

$$\frac{\partial(\rho_p u_p)}{\partial x} + \frac{\partial(\rho_p v_p)}{\partial y} = 0, \tag{2.5}$$

We have the following nomenclature:

(u, v) and (u_p, v_p) denote the velocity components of the fluid and particle phase along the x and y -axis, respectively.

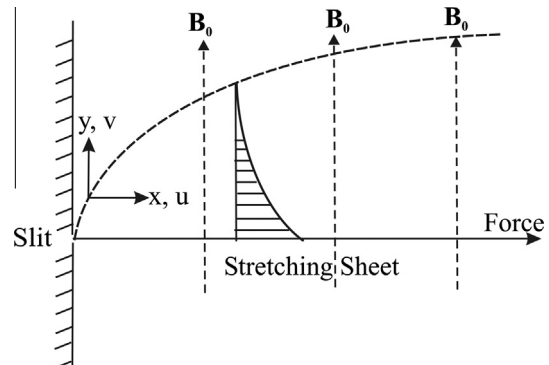


Figure 1 Schematic diagram of the flow.

Further, ρ , $\hat{B}_0\rho_p$, k , N and m are density of the fluid, induced magnetic field, density of particle phase, Stokes constant, number of dust particles and mass of dust particles respectively, and μ is the variable dynamic coefficient of viscosity. In deriving these equations, the Stokesian drag force is considered for the interaction between the fluid and particle phase and the induced magnetic field is neglected. It is also assumed that the external electric field is zero and the electric field due to polarization of charges is negligible.

The boundary conditions applicable to the above problem are

$$\begin{aligned} u &= u_w(x, t) \quad v = v_w(x, t) \quad \text{at } y=0 \\ u &\rightarrow 0, \quad u_p \rightarrow 0, \quad v_p \rightarrow v, \quad \rho_p \rightarrow E\rho, \quad \text{as } y \rightarrow \infty, \quad (0 < E < 1) \end{aligned} \quad (2.6)$$

where $u_w(x, t) = \frac{cx}{1-\alpha t}$ is velocity of sheet, $v_w(x, t) = -\frac{v_0}{\sqrt{1-\alpha t}}$ is suction velocity, v_0 is prescribed constant, E is density ratio and c is stretching rate being a positive constant, α is positive constant which measures the unsteadiness.

Eqs. (2.1)–(2.5) subjected to boundary conditions (2.6), admit self-similar solution in terms of the similarity function f and the similarity variable η as

$$\begin{aligned} u &= \frac{cx}{1-\alpha t} \alpha t f'(\eta), \quad v = -\sqrt{\frac{cv}{1-\alpha t}} f(\eta), \quad u_p = \frac{cx}{1-\alpha t} F(\eta), \\ v_p &= \sqrt{\frac{cv}{1-\alpha t}} G(\eta), \quad \eta = \sqrt{\frac{c}{\eta(1-\alpha t)}} y, \quad \rho_r = H(\eta), \\ \hat{B}_0^2 &= B_0^2(1-\alpha t)^{-\frac{1}{2}}, \quad v = \frac{\mu_0}{\rho}, \end{aligned} \quad (2.7)$$

where prime denotes the differentiation with respect to η .

The variation of the viscosity with temperature is assumed as

$$\mu = \mu_0 e^{-\beta_1 \theta(\eta)} \quad (2.8)$$

where μ_0 is the viscosity at temperature T_w and β_1 is variable viscosity parameter. In general $\beta_1 > 0$ for liquids and $\beta_1 < 0$ for gases.

Substituting set of Eqs. (2.7) and (2.8) into the Eqs. (2.1)–(2.5) and equating the co-efficient of t_0 on both sides of the equations, one can get

$$\begin{aligned} f'''(\eta) + f''(\eta) [f(\eta)e^{\beta_1 \theta(\eta)} - \beta_1 \theta'(\eta)] - A e^{\beta_1 \theta(\eta)} \\ \left[f'(\eta) + \frac{\eta}{2} f''(\eta) \right] - e^{\beta_1 \theta(\eta)} f'(\eta)^2 + \beta H e^{\beta_1 \theta(\eta)} \\ [F(\eta) - f'(\eta)] - M e^{\beta_1 \theta(\eta)} f'(\eta) = 0 \end{aligned} \quad (2.9)$$

$$A \left[F(\eta) + \frac{\eta}{2} F'(\eta) \right] + G(\eta) F'(\eta) + (F(\eta))^2 + \beta [F(\eta) - f'(\eta)] = 0 \quad (2.10)$$

$$\frac{A}{2} \left[G(\eta) + \frac{\eta}{2} G'(\eta) \right] + G(\eta) G'(\eta) + \beta [f(\eta) + G(\eta)] = 0 \quad (2.11)$$

$$H(\eta) F(\eta) + G'(\eta) H(\eta) + G(\eta) H'(\eta) = 0 \quad (2.12)$$

where $\rho_r = \frac{\rho_p}{\rho}$ is relative density, $A = \frac{\rho_p}{\rho}$ is an unsteady parameter which measures unsteadiness, $l = mN/\rho_p$ is mass concentration, $M = \sigma B_0^2/\rho c$ is magnetic parameter and $\beta = \frac{1}{\tau c}$ is fluid particle interaction parameter.

The corresponding boundary conditions are transformed to

$$\begin{aligned} f'(\eta) = 1, f(\eta) = R, \quad \text{at } \eta = 0, \\ f'(\eta) = 0, \quad F(\eta) = 0, \quad G(\eta) = -f(\eta), \quad H(\eta) = E \quad \text{as } \eta \rightarrow \infty \end{aligned} \quad (2.13)$$

where $R = \frac{v_0}{\sqrt{c}}$ is suction parameter.

3. Heat transfer analysis

The governing two dimensional boundary layer equations of energy with variable thermal conductivity are given by

$$\rho c_p \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = \frac{\partial}{\partial y} \left(k^* \frac{\partial T}{\partial y} \right) + \frac{N c_p}{\tau_T} (T_p - T) + \frac{N}{\tau_v} (u_p - u)^2, \quad (3.1)$$

$$N c_m \left[\frac{\partial T_p}{\partial t} + u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} \right] = -\frac{N c_p}{\tau_T} (T_p - T), \quad (3.2)$$

where T and T_p are the temperature of the fluid and dust particle respectively, c_p and c_m are the specific heat of fluid and dust particles respectively, τ_T is the thermal equilibrium time i.e., time taken by the dust cloud to adjust its temperature to that of fluid, τ_v is the relaxation time of the dust particle i.e., the time taken by the dust particle to adjust its velocity to that of fluid. The variation of thermal diffusivity with the dimensionless temperature k^* is assumed to vary as a linear function of temperature in the form

$$k^* = k_0^* (1 + \beta_2 \theta(\eta)), \quad (3.3)$$

where β_2 is a parameter which depends on the nature of the fluid and k_0^* is the value of thermal diffusivity at the temperature T_w .

The solution of Eqs. (3.1) and (3.2) depends on the nature of the prescribed boundary condition. Here, the two types of heating processes are discussed.

3.1. Case-1: variable wall temperature (VWT-Case)

For this heating process, the variable wall temperature is assumed to be a quadratic function of x and is given by

$$\begin{aligned} T = T_w = T_\infty + T_0 \left(\frac{cx^2}{v} \right) (1 - \alpha t)^{-2} \quad \text{at } y = 0, \\ T \rightarrow T_\infty, T_p \rightarrow T_\infty \quad \text{as } y \rightarrow \infty, \end{aligned} \quad (3.4)$$

In order to obtain similarity solution for the temperatures $\theta(\eta)$ and $\theta_p(\eta)$ define dimensionless temperature variables as

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \theta_p(\eta) = \frac{T_p - T_\infty}{T_w - T_\infty}, \quad (3.5)$$

where $T - T_\infty = T_0 \left(\frac{cx^2}{v} \right) (1 - \alpha t)^{-2} \theta(\eta)$.

Using Eqs. (3.3)–(3.5) in the Eqs. (3.1) and (3.2), we get

$$\begin{aligned} \theta''(\eta) + \left[\frac{Pr f(\eta) + \beta_2 \theta'(\eta)}{(1 + \beta_2 \theta(\eta))} \right] \theta'(\eta) - \left[\frac{2Pr f'(\eta) \theta(\eta)}{(1 + \beta_2 \theta(\eta))} \right] \\ - \frac{A Pr}{2} \left[\frac{4\theta(\eta) + \eta \theta'(\eta)}{(1 + \beta_2 \theta(\eta))} \right] + \frac{N}{\rho} \left[\frac{Pra_1}{(1 + \beta_2 \theta(\eta))} \right] [\theta_p(\eta) - \theta(\eta)] \\ + \frac{N}{\rho} \left[\frac{Pr Ec \beta}{(1 + \beta_2 \theta(\eta))} \right] [F(\eta) - f'(\eta)]^2 = 0, \end{aligned} \quad (3.6)$$

$$\begin{aligned} G(\eta) \theta_p'(\eta) + 2F(\eta) \theta_p(\eta) \\ + \frac{A}{2} [4\theta_p(\eta) + \eta \theta_p'(\eta)] + a_1 \gamma [\theta_p(\eta) - \theta(\eta)] = 0, \end{aligned} \quad (3.7)$$

where $Pr = \frac{\mu c_p}{k^*}$ is Prandtl number, $Ec = \frac{v^2}{c_p T_0}$ is the Eckert number, $a_1 = \frac{1}{\tau c}$ is local fluid particle interaction parameters for temperature $\gamma = \frac{c_p}{c_m}$.

Using the Eqs. (3.4) and (3.5), the corresponding boundary conditions for $\theta(\eta)$ and $\theta_p(\eta)$ reduce to the following form:

Table 1 Comparison of numerical solutions of $f(\eta)$, $f'(\eta)$, $f''(\eta)$ with $A = 0$, $M = 3$, $R = 2$, $\beta = 0$, $\beta_1 = 0$.

η	Vajravelu and Nayfeh [19]			Present result		
	$f(\eta)$	$f'(\eta)$	$f''(\eta)$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0.00000	2.000000	1.000000	-3.236068	2.000000	1.000000	-3.236068
0.408163	2.226537	0.266910	-0.863739	2.226537	0.266910	-0.863739
0.816327	2.287002	0.071241	-0.230540	2.287002	0.071241	-0.230540
1.224490	2.303141	0.019015	-0.061534	2.303141	0.019015	-0.061534
1.632653	2.307448	0.005075	-0.016424	2.307449	0.005075	-0.016424
2.040816	2.308598	0.001355	-0.04384	2.308598	0.001355	-0.004384
2.448980	2.308905	0.000362	-0.001170	2.308905	0.000362	-0.001170
2.857143	2.308987	0.000096	-0.000312	2.308987	0.000097	-0.000312
3.265306	2.309009	0.000026	-0.000083	2.309009	0.000026	-0.000083
3.673469	2.309015	0.000007	-0.000022	2.309015	0.000007	-0.000022
4.081633	2.309016	0.000002	-0.000006	2.309016	0.000002	-0.000006
4.489796	2.309017	0.000000	-0.000002	2.309017	0.000000	-0.000002
4.897959	2.309017	0.000000	-0.000001	2.309017	0.000000	0.000000
5.000000	2.309017	0.000000	0.000000	2.3090167	0.000000	0.000000

Table 2 Skin friction coefficient $e^{-\beta_1 f''(0)}$, wall temperature gradient $\theta'(\eta)$ and temperature function $\theta(\eta)$ for different values A , M , β_1 and β_2 .

A	VWT case		VHF case	
	$e^{-\beta_1 f''(0)}$	$\theta'(\eta)$	$e^{-\beta_1 f''(0)}$	$\theta(\eta)$
0.0	-2.643396	-1.206550	-2.852673	0.548101
0.1	-2.684673	-1.267211	-2.864073	0.533465
0.3	-2.771767	-1.374726	-2.892334	0.508316
M				
1.0	-4.314647	-1.761474	-3.193035	0.514254
2.0	-4.638781	-1.746298	-3.479085	0.519482
3.0	-4.926457	-1.734072	-3.732093	0.523799
β_1				
0.0	-3.319485	-1.791234	-2.529445	0.481936
0.3	-4.206107	-1.757306	-2.876099	0.486340
0.5	-4.926457	-1.734072	-3.137902	0.489493
β_2				
0.0	-4.032779	-2.044705	-2.861168	0.466044
0.06	-4.028509	-1.874702	-2.876099	0.486340
0.12	-4.024656	-1.732386	-2.892334	0.508316

$$\begin{aligned} \theta(\eta) &= 1 \quad \text{at } \eta = 0 \\ \theta(\eta) &\rightarrow 0, \quad \theta_p(\eta) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty \end{aligned} \tag{3.8}$$

3.2. Case-2: variable heat flux (VHF-Case)

For this heating process, employ the following variable heat flux boundary conditions:

$$\begin{aligned} \frac{\partial T}{\partial y} &= \frac{q_w(x, t)}{k^*} \quad \text{at } y = 0, \\ T &\rightarrow T_\infty, \quad T_p \rightarrow T_\infty \quad \text{as } y \rightarrow \infty, \end{aligned} \tag{3.9}$$

where $q_w(x, t) = q_{w0} x^2 \left(\frac{t}{v}\right)^{\frac{3}{2}} (1 - \alpha t)^{-\frac{5}{2}}$.

In order to obtain similarity solution for temperature, define the dimensionless temperature variables in VHF case as same in Eq. (3.5) whereas $T_w - T_\infty = \frac{q_{w0}}{k^*} \left(\frac{cx^2}{v(1-\alpha t)}\right) \theta(\eta)$.

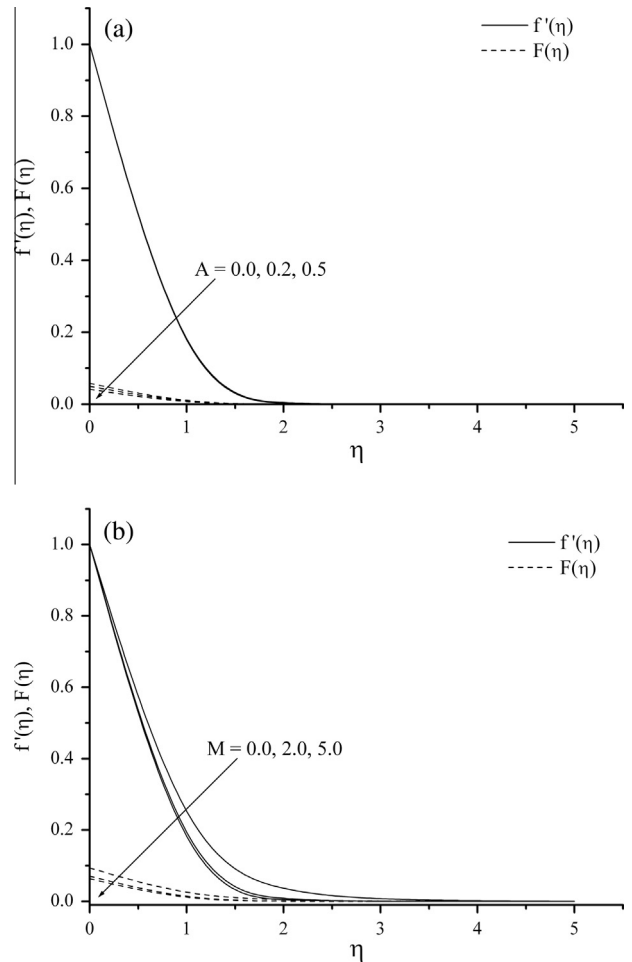


Figure 2 Effect of unsteady parameter A and Magnetic parameter M for fluid and dust phase velocity.

Using dimensionless variable (3.5), the temperature Eqs. (3.1) and (3.2) will take the form

$$\theta''(\eta) + \left[\frac{Prf(\eta) + \beta_2\theta'(\eta)}{(1 + \beta_2\theta(\eta))} \right] \theta'(\eta) - \left[\frac{2Prf'(\eta)\theta(\eta)}{(1 + \beta_2\theta(\eta))} \right] - \frac{APr}{2} \left[\frac{4\theta(\eta) + \eta\theta'(\eta)}{(1 + \beta_2\theta(\eta))} \right] + \frac{N}{\rho} \left[\frac{Pra_1}{(1 + \beta_2\theta(\eta))} \right] [\theta_p(\eta) - \theta(\eta)] + \frac{N}{\rho} \left[\frac{PrEc\beta}{(1 + \beta_2\theta(\eta))} \right] [F(\eta) - f'(\eta)]^2 = 0, \quad (3.10)$$

$$G(\eta)\theta'_p(\eta) + 2F(\eta)\theta_p(\eta) + \frac{A}{2}[4\theta_p(\eta) + \eta\theta'(\eta)] + a_1\gamma[\theta_p(\eta) - \theta(\eta)] = 0, \quad (3.11)$$

The corresponding boundary conditions becomes

$$\theta(\eta) = -1 \quad \text{at} \quad \eta = 0, \quad (3.12)$$

$$\theta(\eta) \rightarrow 0, \quad \theta_p(\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (3.13)$$

The physical quantities of interest in this problem are the skin friction coefficient C_f and the local Nusselt number Nu_x , which are defined as

$$C_f = \frac{\tau_w}{\rho U_w^2}, \quad Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \quad (3.14)$$

where the skin friction τ_w and the heat transfer from the sheet q_w are given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k \left(\frac{\partial T}{\partial y} \right)_{y=0} \quad (3.15)$$

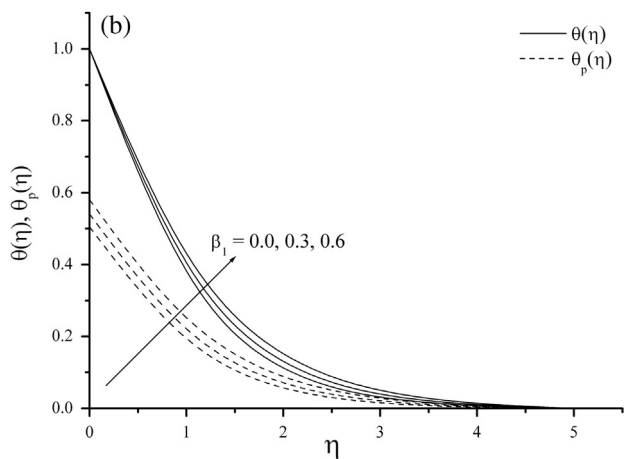
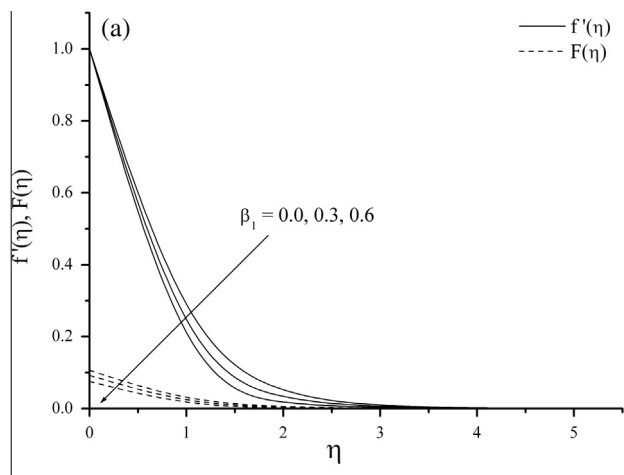


Figure 3 Effect of variable viscosity parameter β_1 on velocity and temperature distribution for fluid and dust phase.

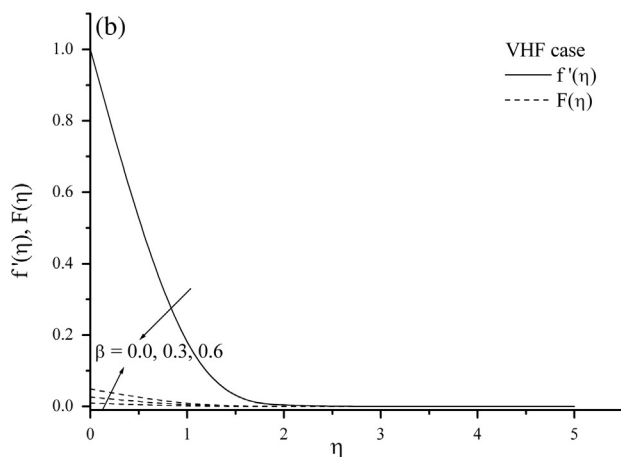
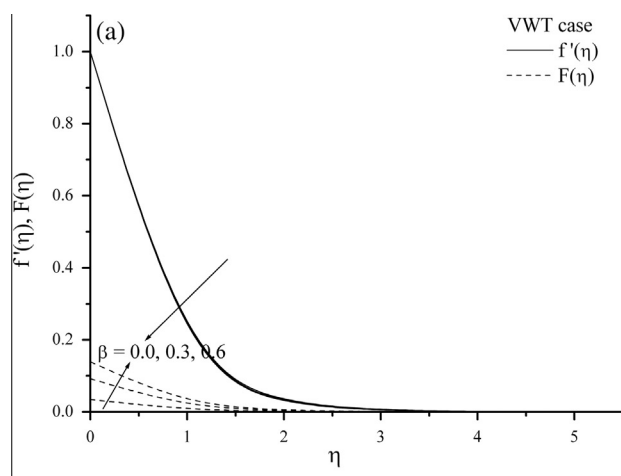


Figure 4 Effect of fluid interaction parameter β on velocity for fluid and dust phase.

Using the non-dimensional variables, we obtain

$$C_f Re_x^{1/2} = e^{-\beta_1} f''(0), \quad Nu_x / Re_x^{1/2} = -\theta'(0) \quad (\text{VWT}), \quad Nu_x / Re_x^{1/2} = -\frac{1}{\theta(0)} \quad (\text{VHF}).$$

4. Numerical solution

The governing partial differential Eqs. (2.1)–(2.5) of continuity and momentum are reduced into non-linear ordinary differential equations using the set of similarity variable (2.7). The obtained equations constitute a highly non-linear coupled boundary value problem of third, first and second order, respectively and are solved numerically by using well known RKF-45 method with the help of symbolic software Maple [22]. To assess the validity and accuracy of the present numerical scheme, computed values of $f(\eta)$, $f'(\eta)$, $f''(\eta)$ for $(A = 0, M = 3, R = 2, \beta_1 = 0)$ are compared with those reported by Vajravelu and Nayfeh [19] and are shown in Table 1. It is observed that a very good agreement has been achieved with their results.

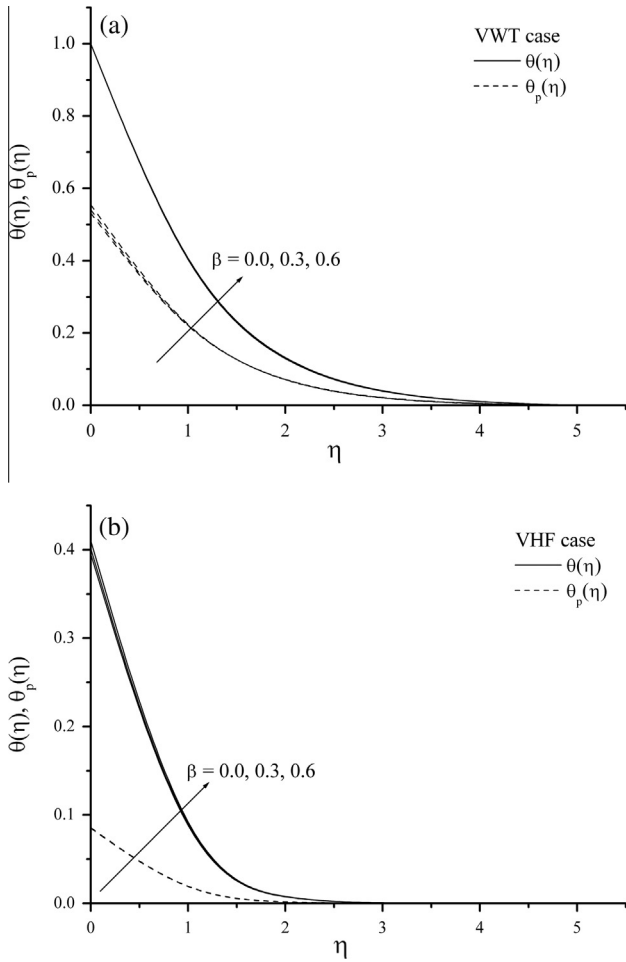


Figure 5 Effect of fluid interaction parameter β on temperature distribution for fluid and dust phase.

5. Results and discussion

By applying similarity analysis to the governing equations and the boundary conditions, the two independent variables are reduced by one; consequently, the governing equations reduce to a system of non-linear ordinary differential equations with the appropriate boundary conditions. The obtained non-linear ordinary differential equations possess no closed form solutions. Therefore, a numerical solution of the problem under consideration is needed. Finally, the system of similarity Eqs. (2.9)–(2.12), and either Eqs. (3.6) and (3.7) for VWT case or (3.10) and (3.11) for VHF case are solved numerically. The numerical solutions are obtained for various values of the parameters such as unsteadiness parameter A , magnetic parameter M , fluid particle interaction parameter β , Prandtl number Pr , Eckert number Ec , viscosity variation parameter β_1 , and thermal diffusivity parameter β_2 . For positive values of β_1 the viscosity of the fluid decreases with an increase in the temperature and this is the case for fluids such as water and lubrication oils, while for negative values of β_1 the viscosity of the fluid increases with an increase in the temperature. On the other hand, the positive values of β_2 mean that the thermal diffusivity increases with an increase in temperature and

this is the case for fluids such as water and air, while for negative values of β_2 , the thermal diffusivity decreases with an increase in the temperature and this is the case for fluids such as lubrication oils. From the above discussions the range of variations of the parameters of the flow β_1 and β_2 can be taken as follows [23]:

- (1) For air $-0.7 \leq \beta_1 \leq 0, 0 \leq \beta_2 \leq 6$.
- (2) For water $0 \leq \beta_1 \leq 0.6, 0 \leq \beta_2 \leq 0.12$.
- (3) For lubricants $0 \leq \beta_1 \leq 3, -0.1 \leq \beta_2 \leq 0$.

It is observed from Table 2 that the skin friction coefficient decreases with increasing values of unsteady parameter A , magnetic parameter M , and variable viscosity parameter β_1 . Here the negative value of $e^{-\beta_1} f''(0)$ means the solid surface exerts a drag force on the fluid. This is due to the development of the velocity boundary layer is caused solely on the stretching plate, whereas increasing effect of thermal diffusivity parameter β_2 is to increase the skin friction coefficient. Nusselt number decreases with increase in unsteady parameter A , whereas it increases with increase in the magnetic parameter M for both VWT and VHF cases. The effect of variable viscosity parameter and thermal diffusivity parameter is to increase the Nusselt number in both the cases.

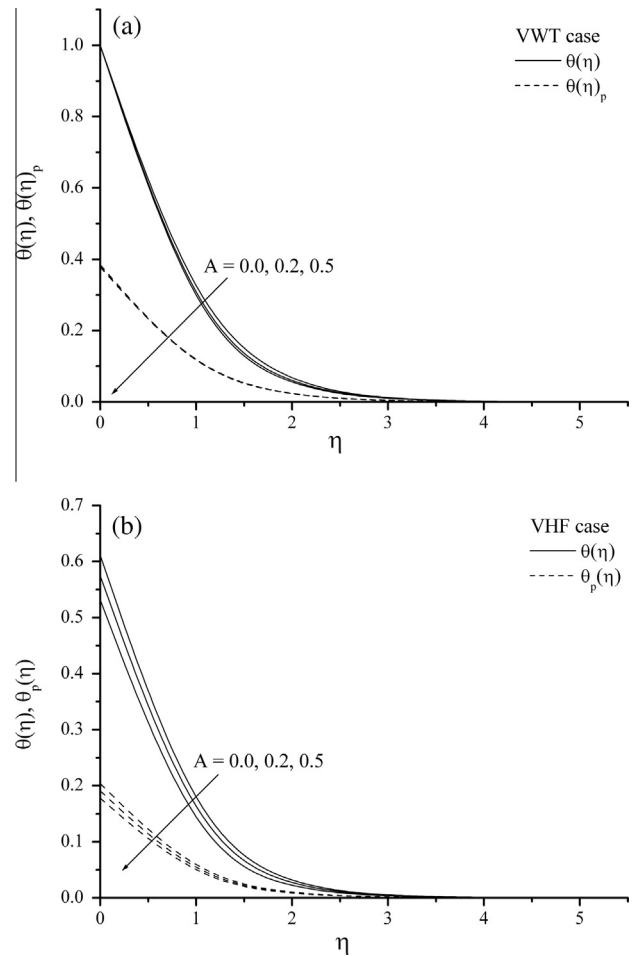


Figure 6 Effect of unsteady parameter A on temperature distribution for fluid and dust phase.

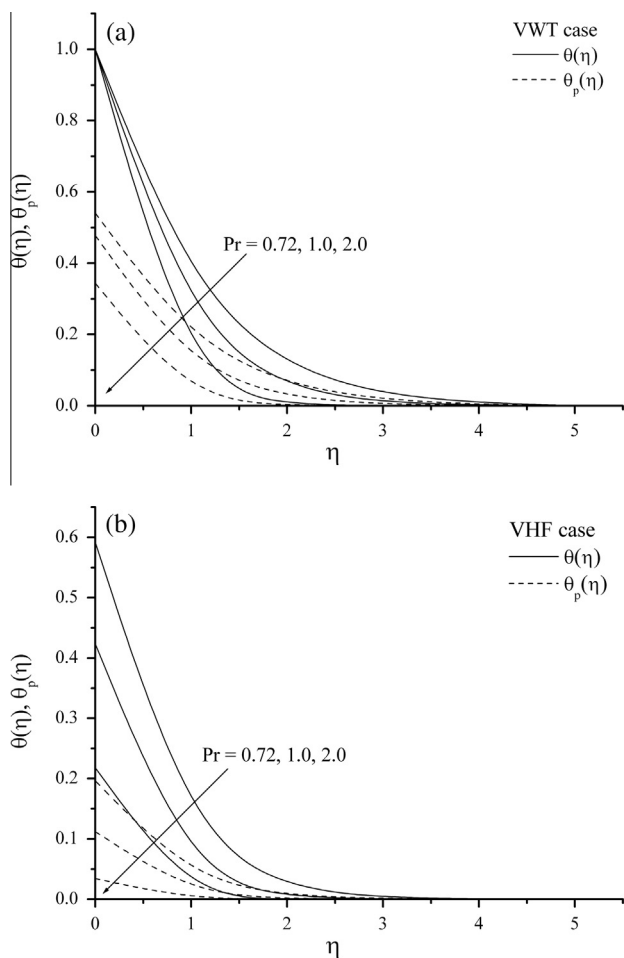


Figure 7 Effect of Prandtl number Pr on temperature distribution for fluid and dust phase.

Fig. 2(a) represents horizontal velocity profile of both fluid and dust particles for various values of A when $Pr = 0.72, R = 2, M = 0.5, N = 0.2,$ and $\beta = 0.5$. From this one can observe that the velocity decreases with the increase of the unsteady parameter A . It is interesting to note that the thickness of boundary decreases with increasing values of A . This is due to the fluid flow caused solely by the stretching sheet.

From Fig. 2(b) it is observed that the velocity decreases with increase of magnetic parameter M . This is because of the fact that the introduction of transverse magnetic field to an electrically conducting fluid gives rise to a resistive type of force, known as Lorentz force. This force has tendency to slow down the motion of fluid in the boundary layer and hence it leads to enhanced deceleration of the flow.

The effect of fluid viscosity parameter β_1 on horizontal velocity profiles $f'(\eta)$ verses η is depicted in Fig. 3(a). It is observed that the horizontal velocity profiles $f'(\eta)$ decrease with an increasing values of fluid viscosity parameter β_1 . This is because of the fact that with an increase in the value of fluid viscosity parameter β_1 decreases the horizontal velocity boundary layer thickness. Physically, this is because a given larger β_1 implies higher temperature difference between the surface and the ambient fluid. The effect of variable viscosity

parameter on temperature is depicted in Fig. 3(b), and the increase in the value of β_1 has tendency to increase the thermal boundary layer thickness. This causes to increase the values of $\theta(\eta)$; thus, increase in the β_1 increases the temperature of the fluid and dust phase.

Fig. 4(a) and (b) shows the effect of fluid particle interaction parameter β on velocity components of the fluid phase velocity $f'(\eta)$ and dust phase velocity $F(\eta)$ for both VWT and VHF cases. It is observed that if β increases we can find the decrease in the fluid phase velocity and increase in the dust phase velocity. Also it reveals that for large values of β , the relaxation time of the dust particle decreases then the velocities of both fluid and dust particle will be the same.

Fig. 5(a) and (b) depicts the effect of fluid particle interaction parameter β on temperature profile for both VWT and VHF cases. The temperature of the fluid and dust phase increases with the increase in the value of β and it indicates that the fluid particles temperature is parallel to the dust particles temperature. Also one can observe that fluid phase temperature is higher than that of dust phase.

Fig. 6(a) and (b) is the graphical representation for temperature distribution of VWT and VHF cases, for different values of unsteady parameter A . It is evident that temperature of fluid and dust particle is found to be decrease with increase of

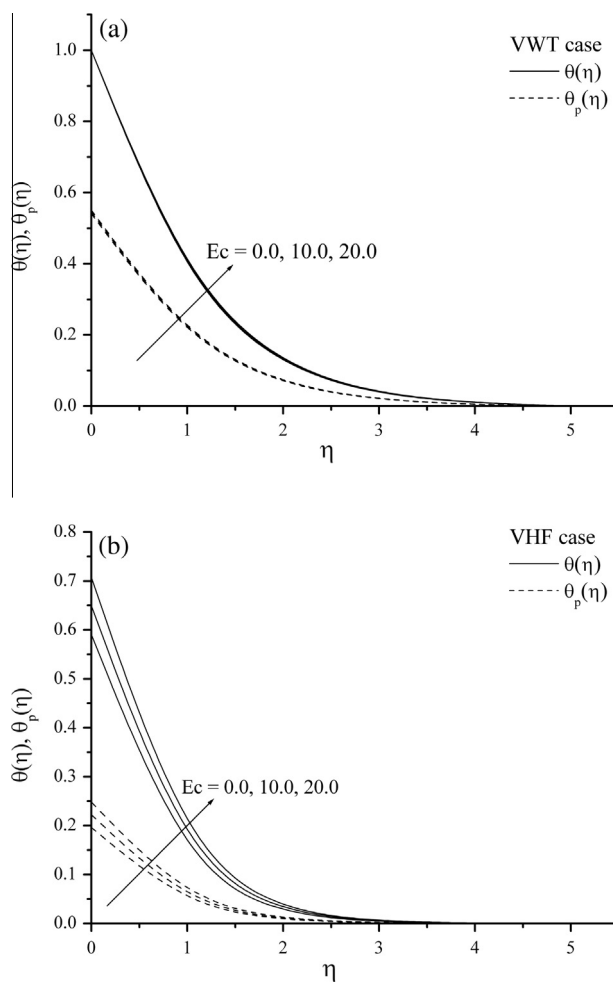


Figure 8 Effect of Eckert number Ec on temperature distribution for fluid and dust phase.

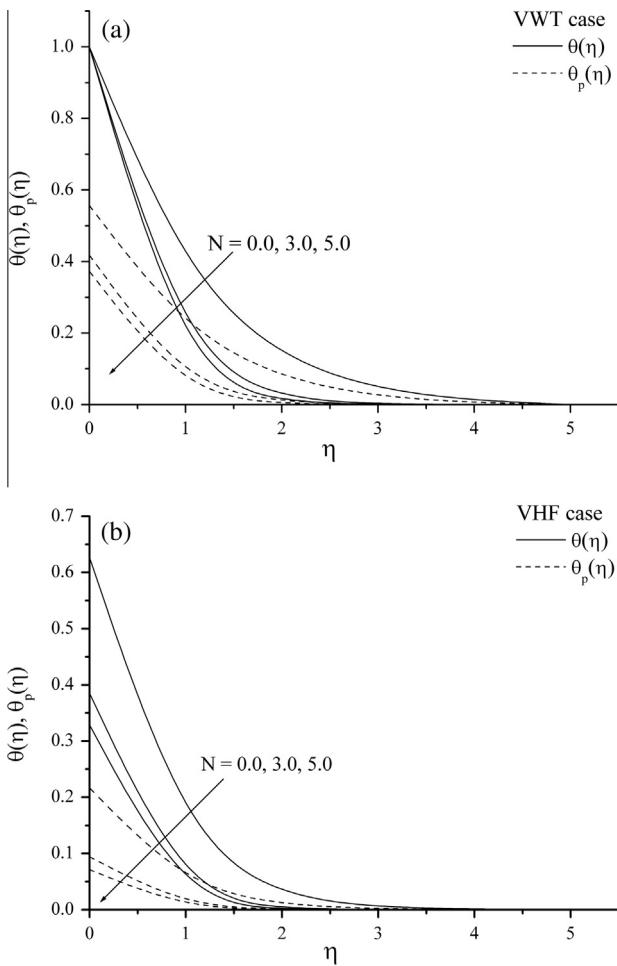


Figure 9 Effect of Number density N on temperature distribution for fluid and dust phase.

unsteady parameter A . Temperature at a point of surface decreases significantly with the increase of A i.e. rate of heat transfer increases with increasing values of unsteady parameter A . Physically, it means that the temperature gradient at the surface increases as A increases, which imply the increase of heat transfer rate $-\theta'(0)$ at the surface. We have used the value $a_1 = 2$ throughout our thermal analysis.

Fig. 7(a) and (b) depicts temperature profiles of $\theta(\eta)$ and $\theta_p(\eta)$ for different values of Pr . We infer from these figures that temperature of fluid and dust particles decreases with the increase in Pr , and this is because of the increase in Prandtl number Pr indicates the increase of the fluid heat capacity or the decrease of the thermal diffusivity, hence causes a diminution of the influence of the thermal expansion to the flow, physically which means that the momentum boundary layer is thicker than the thermal boundary layer. The temperature in both VWT and VHF cases asymptotically approaches to zero in the free stream region.

Fig. 8(a) and (b) indicates the temperature profile of $\theta(\eta)$ and $\theta_p(\eta)$ for VWT and VHF cases respectively. Here the effect of increase in the Eckert number values greatly affects the temperature of the fluid and dust phase. The temperature of the fluid and dust phase increases as the Eckert number Ec increases for both the cases VWT and VHF. This is due to fact

that the heat energy is stored in the liquid due to frictional heating.

Fig. 9(a) and (b) shows the temperature profiles of $\theta(\eta)$ and $\theta_p(\eta)$ for different values of Number density N for VWT and VHF cases respectively. From these figures, it is observed that the temperature of fluid and dust phases decreases with the increase of N .

Fig. 10(a) and (b) shows that the variation of thermal conductivity affects temperature profiles. The increase in the value of β_2 increases the temperature of fluid and dust phase; hence, the variation of thermal conductivity cannot be neglected. Moreover, the rise in the magnitude of the temperature is quite significant, showing that the volume rate of flow at a section perpendicular to the plate increases with the increase in β_2 .

Fig. 11(a) and (b) reveals the effect of magnetic parameter M and variable viscosity parameter β_1 on transverse velocity $G(\eta)$. From these figures, it is noticed that the increase in magnetic parameter and viscosity parameter decreases the velocity profile $G(\eta)$. Since the numerical values of the transverse velocity at different values of parameters are negative, the arrow mark is shown upwards in the graph.

Fig. 12(a) and (b) displays the effect of magnetic parameter M and variable viscosity parameter β_1 on particle density

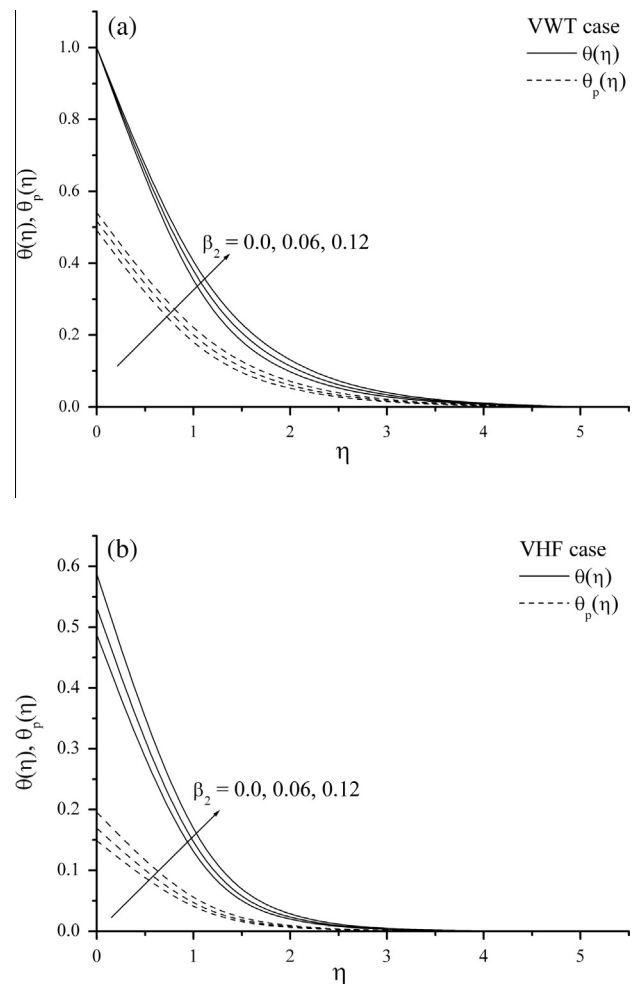


Figure 10 Effect of thermal diffusion parameter β_2 on temperature distribution for fluid and dust phase.

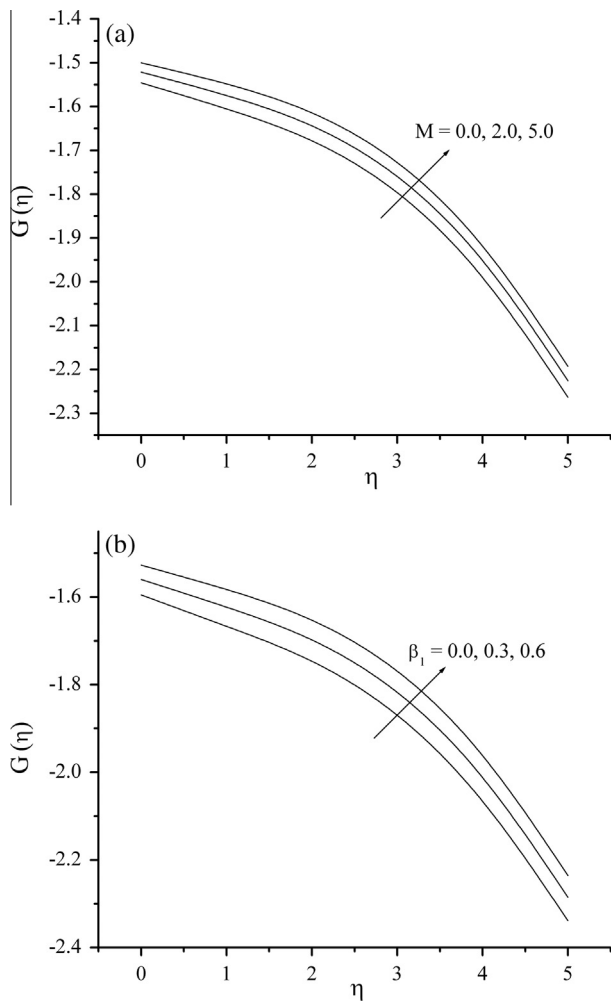


Figure 11 Effect of Magnetic parameter M on $H(\eta)$ and $G(\eta)$.

$H(\eta)$. These figures infer that particle density $H(\eta)$ increases as magnetic and variable viscosity parameters increase, i.e. the particle density is maximum at the surface of the stretching sheet.

6. Conclusions

The effect of variable viscosity and thermal diffusivity on MHD boundary layer flow of dusty fluid flow over a stretching sheet has been investigated. The highly non-linear momentum Eqs. (2.1)–(2.5) and heat transfer boundary layer Eqs. (3.1) and (3.2) are reduced into coupled ordinary differential equations using similarity transformations. Resultant coupled ordinary differential Eqs. (2.9)–(2.12) and (3.6) and (3.7) for VWT case and (3.10) and (3.11) for VHF case have been solved numerically by RK45 method [22]. The results are analyzed for the situation when stretching boundary is prescribed by non-isothermal variable wall temperature (VWT) and variable heat flux (VHF) which varies quadratically with the flow directional coordinate x . The effect of various physical parameters such as unsteady parameter A , Prandtl number Pr , Eckert number Ec , magnetic parameter M , variable viscosity parameter β_1 and thermal diffusivity parameter β_2 on heat transfer characteristics is discussed for different numerical values with

the help of Table 3. Some of the interesting observations of the study are listed as follows:

- Increase in unsteady parameter and magnetic parameter, decreases the velocity components of both fluid and dust phases.
- The increasing effect of fluid interaction parameter decreases the fluid phase velocity and increases the dust phase velocity.
- Increase in the variable viscosity parameter decreases the fluid and dust phase velocities whereas it increases the temperature profiles of both the phases.
- Effect of unsteady parameter decreases the temperature profiles of fluid and dust phase for both the cases of VWT and VHF.
- Effect of thermal diffusivity parameter increases the temperature profiles of fluid and dust phase for both the cases of VWT and VHF.
- Always the rate of heat transfer $\theta'(0)$ is negative and $\theta(0)$ is positive.
- $\theta'(0)$ and $\theta(0)$ increase with the increasing values of fluid interaction parameter, variable viscosity parameter, thermal diffusivity parameter and Eckert number.

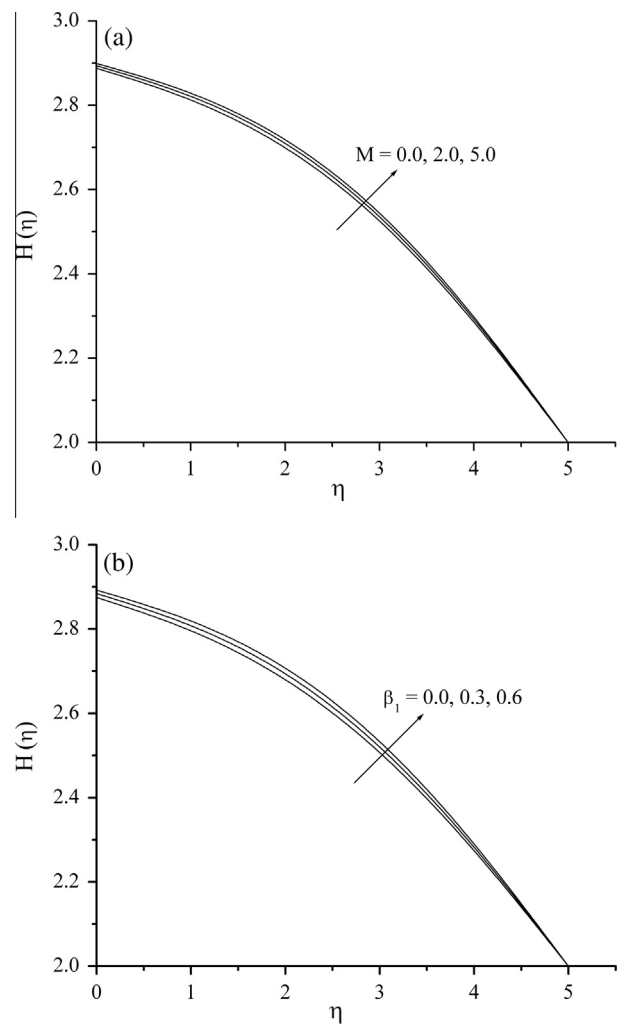


Figure 12 Effect of variable viscosity parameter β_1 on $H(\eta)$ and $G(\eta)$.

Table 3 Values of wall temperature gradient $\theta'(0)$ for (VWT Case) and wall temperature function $\theta(0)$ (for VHF Case) for different values of β , A , Pr , Ec , N , β_1 and β_2 .

β	A	Pr	Ec	N	β_1	β_2	$\theta'(0)$ (VWT case)	$\theta(0)$ (VHF case)
0.1	0.5	0.72	0.5	0.5	0.5	0.12	-1.282118	0.531004
0.5							-1.291514	0.529930
1							-1.295391	0.529643
0.1	0	0.72	0.5	0.5	0.5	0.12	-2.806692	0.605924
	0.2						-2.874686	0.570786
	0.5						-2.918722	0.529619
0.1	0.5	0.72	0.5	0.5	0.5	0.12	-1.274655	0.529619
		1.0					-1.640112	0.390520
		2.0					-2.847079	0.207911
0.1	0.5	0.72	0	0.5	0.5	0.12	-2.919233	0.528990
			0.5				-2.915406	0.530561
			2.0				-2.911579	0.535272
0.1	0.5	0.72	0.5	0.5	0.5	0.12	-2.992918	0.502526
				1.5			-3.220744	0.436514
				3.0			-3.521202	0.3746170
0.1	0.5	0.72	0.5	0.5	0.1	0.12	-1.514773	0.282630
					0.3		-1.502429	0.282838
					0.5		-1.489785	0.283047
0.1	0.5	0.72	1	1	0.5	0.0	-1.854711	0.261188
						0.06	-1.713408	0.267973
						0.12	-1.489785	0.282630

- The variable thermal conductivity has an impact in enhancing the skin friction coefficient; hence, fluids with less thermal conductivity may be opted for effective cooling.
- The increase in Prandtl number decreases the thermal boundary layer thickness.
- If $M = 3$, $R = 2$, $\beta = 0$, $\beta_1 = 0$ and $A \rightarrow 0$, then our results coincide with the results reported by Vajravelu and Nayfeh [19].

Acknowledgment

We express our thanks to University Grant Commission, New Delhi, for financial support to pursue this work under a Major Research Project (F.No.36-147/2008/(SR)/dated: 26-03-2009).

References

- [1] Chakrabarti A, Gupta AS. Hydromagnetic flow and heat transfer over a stretching sheet. *Quart J Appl Math* 1979;37:73–8.
- [2] Sakiadis BC. Boundary layer behaviour on continuous solid surface; I Boundary-layer equations for two-dimensional and ax symmetric flow. *AIChE J* 1961;7:26–8.
- [3] Chiam TC. Hydrodynamic flow over a surface stretching with a power law velocity. *Int J Eng Sci* 1995;33:429–35.
- [4] Tsou FK, Sparrow EM, Goldstein RJ. Flow and heat transfer in the boundary layer on a continuous moving surface. *Int J Heat Mass Transfer* 1967;10:219–35.
- [5] Crane LJ. Flow past a stretching sheet. *ZAMP* 1970;21:645–7.
- [6] Grubka LJ, Bobba KM. Heat transfer characteristics of a continuous stretching surface with variable temperature. *Int J Heat Mass Transfer* 1985;107:248–50.

- [7] Elbashebsy EMA. Heat transfer over a stretching surface with variable surface heat flux. *J Phys D Appl Phys* 1998;31:1951–4.
- [8] Elbashebsy EMA, Bazid MAA. Heat transfer over an unsteady stretching surface. *Heat Mass Transfer* 2004;41:1–4.
- [9] Sharidan S, Mahmoodand T, Pop I. Similarity solutions for the unsteady boundary layer flow and heat transfer due to a stretching sheet. *Int J Appl Mech Eng* 2006;11(3):647–54.
- [10] Herwing H, Gersten K. The effect variable properties on laminar boundary layer flow. *Warme Stoffubertrag* 1986;20:47–57.
- [11] Lai FC, Kulacki FA. The effect of variable viscosity on convective heat transfer along a vertical surface in a saturated porous medium. *Int J Heat Mass Transfer* 1990;33:1028–31.
- [12] Pop I, Gorla RSR, Rashidi M. The effect of variable viscosity on flow and heat transfer to a continuous moving flat plate. *Int J Eng Sci* 1992;30(1):1–6.
- [13] Abel MS, Khan SK, Prasad KV. Study of visco-elastic fluid and heat transfer over a stretching sheet with variable viscosity. *Int J Non-linear Mech* 2002;37:81–8.
- [14] McCormack PD, Crane LJ. *Physical fluid dynamics*. New York: Academic Press; 1973.
- [15] Salem AM. Variable viscosity and thermal conductivity effects on MHD flow and heat transfer in viscoelastic fluid over a stretching sheet. *Phys Lett A* 2007;369(4):315–22.
- [16] Chiam TC. Heat transfer in a fluid with variable thermal conductivity over a linearly stretching sheet. *Acta Mech* 1998;129:63–72.
- [17] Pantokratoras A. Further results on the variable viscosity on the flow and heat transfer to a continuous moving flat plate. *Int J Eng Sci* 2004;42:1891–6.
- [18] Mukhopadhyay S, Layek GC. Effect of thermal radiation and variable fluid viscosity on free convective and heat transfer past a porous stretching surface. *Int J Heat Mass Transfer* 2008;2167–78.
- [19] Vajravelu K, Nayfeh J. Hydromagnetic flow of a dusty fluid over a stretching sheet. *Int J Nonlinear Mech* 1992;27(6):937–45.
- [20] Gireesha BJ, Manjunatha S, Bagewadi CS. Unsteady hydromagnetic boundary layer flow and heat transfer of dusty fluid over a stretching sheet. *Afrika Metamet* 2012;23(2):229–41.
- [21] Gireesha BJ, Ramesh GK, Subhas Abel M, Bagewadi CS. Boundary layer flow and heat transfer of a dusty fluid flow over a stretching sheet with non-uniform heat source/sink. *Int J Multiphase Flow* 2011;37(8):977–82.
- [22] Aziz Abdul. A similarity solution for laminar thermal boundary layer over a flat plate with a convective surface boundary condition. *Commun Nonlinear Sci Numer Simul* 2009;14:1064–8.
- [23] Schlichting H. *Boundary layer theory*. New York: McGraw-Hill; 1968.



Manjunatha S. was born in 1982 and brought up in his district Chitradurga, Karnataka, India. He received his Master degree in the year 2008 and he joined as a research scholar in 2009 in the Department of Mathematics, Kuvempu University. He has worked as a research fellow for UGC project **Entitled Study on Boundary Layer Flow of UN Steady Dusty Fluid from the year 2009–2012**. He is awarded his Ph.D. degree in the year 2014 in Mathematics from Kuvempu University, Shimoga, India. His field of interest covers the areas of the application of boundary layer flows, Heat/Mass transfer and Dusty fluid flow problems. He has authored and co-authored 12 national and international research papers. He has attended/presented the papers in 8 International/National conferences.



B.J. Gireesha was born in Kolalu village, Karnataka, India, on 1st June 1974 and received his Master degree in Mathematics (1997), M.Phil. (1999) and Ph.D. (2002) in Fluid Mechanics from Kuvempu University, Shimoga, India. Currently he is working as a faculty in the Department of Mathematics, Kuvempu University. He has authored and co-authored 3 books, 107 national and international research papers, 9 conference papers and worked as an editor for 2 conference

proceedings. He has completed 2 major research projects and one

minor project sponsored by UGC and DST, New Delhi, India. He has attended/presented the papers in 26 International/National conferences. He is a member of several bodies, Editorial Board member and reviewer for several National/International research Journals. His research interests include the areas of Fluid mechanics particularly, in boundary layer flows, Newtonian/non-Newtonian fluids, heat and mass transfer. His research interest also covers the Nanofluid flow problems, existence and uniqueness of solutions and other related matters. 7 students got Ph.D., degree and 11 students were awarded M.Phil., degree under his supervision. Presently he is guiding 8 students for Ph.D.