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Theta-function identities of level 8 and its application to partitions

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Abstract
S. Ramanujan has recorded many theta-function identities in his notebooks and 'Lost' notebook. Inspired by the works of Ramanujan, recently M. Somos discovered several new theta-function identities using PARI/GP scripts without offering the proof, which are analogous to Ramanujan's theta-function identities. In this paper, we give proofs for some theta-function identities of level 8 discovered by Somos. Furthermore, we extract some partition identities from them.

Keywords Theta-functions · Colored partitions

Mathematics Subject Classification Primary 33C05 · 11P83; Secondary 11F20 · 11F27

1 Introduction

Throughout this paper, let $|q| < 1$ and use the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

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In Chapter 16 of his second notebook [1, p. 34], [3, p. 197], Ramanujan developed the theory of theta-function and his theta-function is defined as follows

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1. \tag{1}$$

The important special cases of $f(a,b)$ [1, p. 36] are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left(-q;q^2\right)_\infty^2 \left(q^2;q^2\right)_\infty, \tag{2}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}, \tag{3}$$

$$f(-q) := f\left(-q,-q^2\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty. \tag{4}$$

Also after Ramanujan, define

$$\chi(-q) := \left(q;q^2\right)_\infty.$$

The infinite product representations in (2)–(4) arise from the well known Jacobi's triple product identity [1]

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1.$$

For convenience, we denote $f(-q^n)$ by f_n for any positive integer n and it is easy to see that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \varphi(-q) = \frac{f_1^2}{f_2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \quad \psi(-q) = \frac{f_1 f_4}{f_2},$$
$$f(q) = \frac{f_2^3}{f_1 f_4}, \quad \chi(q) = \frac{f_2^2}{f_1 f_4} \quad \text{and} \quad \chi(-q) = \frac{f_1}{f_2}. \tag{5}$$

Note that, if $q = e^{2\pi i\tau}$ then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind $\eta-$function for $\mathrm{Im}(\tau) > 0$. The theta-function identity which relates $f(-q)$ to $f(-q^n)$ is called theta-function identity of level n. Ramanujan recorded several identities which involve $f(-q)$, $f(-q^2)$, $f(-q^n)$ and $f(-q^{2n})$ in his second notebook [3] and 'Lost' notebook [4]. Somos [5] recently used computer to discover many theta-function identities of different levels. He has a large list of eta-product identities and runs PARI/GP scripts to check his identities. Furthermore, Yuttanan [8] has proved certain Somos's theta-function identities of different levels, by using Ramanujan's modular equations and deduced certain interesting partition identities from them. Vasuki and Veeresha [7] have proved Somos's theta-function identities of level fourteen and Srivatsa Kumar and Veeresha [6] have obtained partition identities for these theta-function identities. Also Mahadeva Naika [2] has obtained similar type of theta function identities on Ramanujan's continued fraction. The purpose of this paper is to prove some of these identities of level 8 and to establish certain combinatorial interpretations of our results. In Somos's identities of level 8, we obtain the arguments in $f(-q)$, $f(-q^2)$, $f(-q^4)$ and $f(-q^8)$, namely $-q$, $-q^2$, $-q^4$ and $-q^8$ all have exponents dividing eight, which is thus equal to the 'level' of the identity eight. Some of Somos's identities are the simple consequences of others and we record only few of them. Our proofs use nothing more than theta-function identities. However our proofs are more elementary and can be extended to prove the modular equations. The results that we are obtaining in Sect. 2

are rich with applications to colored partitions. Further in Sect. 3, we establish combinatorial interpretations for few results.

To prove our main results, we require the following basic results

Lemma 1 *We have*

$$\varphi(q)+\varphi(-q)=2\varphi\left(q^4\right), \tag{6}$$

$$\varphi(q)-\varphi(-q)=4q\psi\left(q^8\right), \tag{7}$$

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2\left(q^2\right), \tag{8}$$

$$\varphi^2(q)-\varphi^2(-q)=8q\psi^2\left(q^4\right), \tag{9}$$

$$\varphi^4(q)-\varphi^4(-q)=16q\psi^4\left(q^2\right), \tag{10}$$

$$\varphi(q)\varphi(-q)=\varphi^2\left(-q^2\right), \tag{11}$$

$$\psi(q)\psi(-q)=\psi(q^2)\varphi\left(-q^2\right) \tag{12}$$

and

$$\varphi(q)\psi\left(q^2\right)=\psi^2(q). \tag{13}$$

Proof For proof see Entry 25 of Chapter 16 [1]. □

2 Main results

Theorem 1 *We have*

$$\varphi(q)\varphi^3(-q)=\varphi^4\left(-q^4\right)-4q\psi^4\left(-q^2\right).$$

Proof Changing q to $-q^2$ in (13), squaring on both sides and then multiplying throughout by $4q$, we get

$$4q\psi^4\left(-q^2\right)=4q\varphi^2\left(-q^2\right)\psi^2\left(q^4\right). \tag{14}$$

Multiplying (11) throughout by $\varphi^2(-q)$, we obtain

$$\varphi(q)\varphi^3(-q)=\varphi^2(-q)\varphi^2\left(-q^2\right). \tag{15}$$

On adding (14) and (15), we obtain

$$\varphi(q)\varphi^3(-q)+4q\psi^4\left(-q^2\right)=\varphi^2\left(-q^2\right)\left[\varphi^2(-q)+4q\psi^2\left(q^4\right)\right]. \tag{16}$$

Subtracting (8) from (9), we obtain

$$\varphi^2(-q)+4q\psi^2\left(q^4\right)=\varphi^2\left(q^2\right). \tag{17}$$

Using (17) in the right-hand side of (16) and then employing (11), we complete the proof. □

Theorem 2 *We have*

$$\chi^4(-q)=\frac{\chi^4\left(-q^4\right)}{\chi^6\left(-q^2\right)}-\frac{4q}{\chi^6\left(-q^2\right)\chi^4\left(q^2\right)}.$$

Proof Using (5) in (7) and (8) and then dividing throughout by f_2^2, we obtain

$$\chi^4(q)+\chi^4(-q)=2\frac{\chi^4\left(-q^4\right)}{\chi^6\left(-q^2\right)} \tag{18}$$

and

$$\chi^4(q)-\chi^4(-q)=\frac{8qf_8^2}{f_2^2\chi^2\left(-q^4\right)}.$$

Multiplying and dividing by f_4^2 in the right-hand side of the above, we obtain

$$\chi^4(q)-\chi^4(-q)=\frac{8q}{\chi^2\left(-q^2\right)\chi^4\left(-q^4\right)}. \tag{19}$$

From Entry 24 [1, p. 39], we have

$$\chi\left(-q^2\right)=\chi(q)\chi(-q). \tag{20}$$

Employing (20) in (19), we deduce that

$$\chi^4(q)-\chi^4(-q)=\frac{8q}{\chi^6\left(-q^2\right)\chi^4\left(q^2\right)}. \tag{21}$$

Subtracting (21) from (18), we obtain the result. □

Theorem 3 *We have*

$$\frac{\varphi^2(-q)}{\psi^2\left(q^4\right)}-4q\frac{\varphi^2(-q)}{\varphi^2\left(q^2\right)}+16q=\chi^8(q)\chi^8(-q^2).$$

Proof It follows from (8) and (9) that

$$\frac{\varphi^2(-q)}{q\psi^2\left(q^4\right)}-4\frac{\varphi^2(-q)}{\varphi^2\left(q^2\right)}+16=\frac{8\varphi^2(-q)}{\varphi^2(q)-\varphi^2(-q)}-\frac{8\varphi^2(-q)}{\varphi^2(q)+\varphi^2(-q)}+16.$$

Now using (10) in the right-hand side of the above, we deduce that

$$\begin{aligned}\frac{\varphi^2(-q)}{q\psi^2\left(q^4\right)}-4\frac{\varphi^2(-q)}{\varphi^2\left(q^2\right)}+16&=\frac{16\varphi^4(-q)}{\varphi^4(q)-\varphi^4(-q)}+16\\&=\frac{\varphi^4(-q)+16q\psi^4\left(q^2\right)}{q\psi^4\left(q^2\right)}\\&=\frac{\varphi^4(q)}{q\psi^4\left(q^2\right)}\\&=\frac{\chi^8(q)\chi^8\left(-q^2\right)}{q}.\end{aligned}$$

This completes the proof. □

Theorem 4 *We have*

$$f^3(-q)=\psi(q)\{\varphi^2\left(q^2\right)-4q\psi^2\left(q^4\right)\}.$$

Proof From (8) and (9), we have

$$\varphi^2\left(q^2\right)=\frac{\varphi^2(q)+\varphi^2(-q)}{2} \tag{22}$$

and

$$4q\psi^2\left(q^4\right)=\frac{\varphi^2(q)-\varphi^2(-q)}{2}. \tag{23}$$

Subtracting (23) from (22) and then multiplying throughout by $\psi(q)$, we find that

$$\psi(q)\left\{\varphi^2\left(q^2\right)-4q\psi^2\left(q^4\right)\right\}=\psi(q)\varphi^2(-q)=f^3(-q),$$

where we have used Entry 24(ii) of Chapter 16 [1]. This completes the proof. □

Theorem 5 *We have*

$$\varphi^4(-q)+8q\psi^4\left(q^2\right)=\varphi^4\left(-q^2\right)+32q^2\psi^4\left(q^4\right).$$

Proof Multiplying (10) throughout by 2 and then replacing q by q^2, we obtain

$$32q^2\psi^4\left(q^4\right)=2\varphi^4\left(q^2\right)-2\varphi^4\left(q^2\right). \tag{24}$$

From (10), we have

$$8q\psi^4\left(q^2\right)=\frac{\varphi^4(q)-\varphi^4(-q)}{2}. \tag{25}$$

On subtracting (24) from (25), we obtain

$$8q\psi^4\left(q^2\right)-32q^2\psi^4\left(q^4\right)=\frac{\varphi^4(q)-\varphi^4(-q)}{2}-2\varphi^4\left(q^2\right)+2\varphi^4\left(-q^2\right). \tag{26}$$

Replacing q by q^2 in (11), we have

$$\varphi\left(q^2\right)\varphi\left(-q^2\right)=\varphi^2\left(-q^4\right). \tag{27}$$

Squaring on both sides of (8), and then using (27) in the resulting identity, we obtain

$$2\varphi^4\left(q^2\right)=\frac{1}{2}\left\{\varphi^4(q)+\varphi^4(-q)+2\varphi^4\left(-q^2\right)\right\}.$$

Using this in the right-hand side of (26), we complete the proof. □

Theorem 6 *We have*

$$\varphi^4(q)+\varphi^4(-q)=2\varphi^4\left(q^2\right)+32q^2\psi^4\left(q^4\right).$$

Proof Adding $2\varphi^4\left(q^2\right)$ on both sides of (24), we obtain

$$2\varphi^4\left(q^2\right)+32q^2\psi^4\left(q^4\right)=4\varphi^4\left(q^2\right)-2\varphi^4\left(-q^2\right). \tag{28}$$

Squaring (8) and (11), we have

$$4\varphi^4\left(q^2\right)=\varphi^4(q)+\varphi^4(-q)+2\varphi^2(q)\varphi^2(-q) \tag{29}$$

and

$$\varphi^2(q)\varphi^2(-q)=\varphi^4\left(-q^2\right), \tag{30}$$

respectively. Employing (30) in (29) and then using the resulting identity in (28), we obtain the required result. □

Theorem 7 *We have*

$$32q\psi^4\left(q^2\right)\left\{\varphi^4\left(q^2\right)+16q^2\psi^4\left(q^4\right)\right\}=\varphi^8(q)-\varphi^8(-q).$$

Proof After rewriting, Theorem 2.6, we have

$$\varphi^4\left(q^2\right)+16q^2\psi^4\left(q^4\right)=\frac{1}{2}\left\{\varphi^4(q)+\varphi^4(-q)\right\}$$

Multiplying the above throughout by $32q\psi^4\left(q^2\right)$, we have

$$32q\psi^4\left(q^2\right)\left\{\varphi^4\left(q^2\right)+16q^2\psi^4\left(q^4\right)\right\}=16q\psi^4\left(q^2\right)\left\{\varphi^4(q)+\varphi^4(-q)\right\}.$$

Using (10) in right hand-side of the above, we complete the proof. □

Theorem 8 *We have*

$$\frac{\chi^4(-q)}{\chi^4(q)}=1-\frac{8q}{\chi^{12}\left(-q^2\right)}+\frac{32q^2}{\chi^8\left(-q^2\right)\chi^8\left(-q^4\right)}.$$

Proof From Theorem 2.5, we have

$$\varphi^4(-q)=\varphi^4\left(-q^2\right)-8q\psi^4\left(q^2\right)+32q^2\psi^4\left(q^4\right).$$

On dividing the above throughout by $\varphi^4\left(-q^2\right)$, we obtain

$$\frac{\varphi^4(-q)}{\varphi^4\left(-q^2\right)}=1-8q\frac{\psi^4\left(q^2\right)}{\varphi^4\left(-q^2\right)}+32q^2\frac{\psi^4\left(q^4\right)}{\varphi^4\left(-q^2\right)}.$$

Employing (5) in the above and after some rearrangement of terms, we complete the proof. □

Theorem 9 *We have*

$$\varphi(q)\psi^2(q)+\varphi(-q)\psi^2(-q)=2\varphi^2\left(q^2\right)\psi\left(q^2\right).$$

Proof Replacing q by $-q$ in (13) and then multiplying the resulting equation by $\varphi(-q)$, we obtain

$$\varphi^2(-q)\psi\left(q^2\right)=\varphi(-q)\psi^2(-q). \tag{31}$$

Again on multiplying (13) by $\varphi(q)$, we have

$$\varphi^2(q)\psi\left(q^2\right)=\varphi(q)\psi^2(q). \tag{32}$$

On adding (31) and (32), we obtain

$$\varphi(q)\psi^2(q)+\varphi(-q)\psi^2(-q)=\psi\left(q^2\right)\left(\varphi^2(q)+\varphi^2(-q)\right).$$

Using (8) in the right-hand side of the above, we complete the proof. □

Theorem 10 *We have*

$$\varphi(-q)(\varphi^4(-q)+\varphi^4\left(-q^2\right)+16q\psi^4\left(q^2\right))=2\varphi(q)\varphi^4\left(-q^4\right).$$

Proof On squaring (11), we find that

$$\varphi^2(q)\varphi^2(-q)=\varphi^4\left(-q^2\right). \tag{33}$$

On adding (10) and (33), we get

$$\begin{aligned}\varphi^4\left(-q^2\right)+\varphi^4(-q)+16q\psi^4\left(q^2\right)&=\varphi^4(q)+\varphi^2(q)\varphi^2(-q)\\&=\varphi^2(q)\left\{\varphi^2(q)+\varphi^2(-q)\right\}.\end{aligned}$$

Now multiplying the above throughout by $\varphi(-q)$ and then using (8) in the right-hand side of the resulting identity, we see that

$$\varphi(-q)\left\{\varphi^4\left(-q^2\right)+\varphi^4(-q)+16q\psi^4\left(q^2\right)\right\}2\varphi(q)\varphi^2\left(q^2\right)\varphi^2\left(-q^2\right).$$

Replacing q by q^2 in (33) and then using the resulting identity in right-hand side of the above, we complete the proof. □

Theorem 11 *We have*

$$\psi^2\left(q^2\right)\{\varphi^2\left(q^2\right)+4q\psi^2\left(q^4\right)\}=\psi^4(q).$$

Proof On squaring (13), we have

$$\varphi^2(q)\psi^2\left(q^2\right)=\psi^4(q). \tag{34}$$

Adding (22) and (23), we get

$$\begin{aligned}\varphi^2\left(q^2\right)+4q\psi^2\left(q^4\right)&=\frac{\varphi^2(q)+\varphi^2(-q)}{2}+\frac{\varphi^2(q)-\varphi^2(-q)}{2}\\&=\frac{\psi^4(q)}{\psi^2\left(q^2\right)},\end{aligned}$$

where we have used (34). This completes the proof. □

Theorem 12 *We have*

$$q(\chi^8(q)-\chi^8(-q))=\chi^{16}\left(q^2\right)-\chi^8\left(-q^4\right).$$

Proof From (18) and (19), we have

$$\chi^4(q)+\chi^4(-q)=2\frac{\chi^4\left(-q^4\right)}{\chi^6\left(-q^2\right)}$$

and

$$\chi^4(q)-\chi^4(-q)=\frac{8q}{\chi^2\left(-q^2\right)\chi^4\left(-q^4\right)}.$$

Multiplying the above two identities, we see that

$$\chi^8(q)-\chi^8(-q)=\frac{16q}{\chi^8\left(-q^2\right)}. \tag{35}$$

Replacing q by q^2 in (20), we have

$$\chi\left(-q^4\right)=\chi\left(q^2\right)\chi\left(-q^2\right). \tag{36}$$

Now consider

$$\begin{aligned}\chi^{16}\left(q^2\right)-\chi^8\left(-q^4\right)=&\chi^{16}\left(q^2\right)-\chi^8\left(-q^2\right)\chi^8\left(q^2\right)\\=&\chi^8\left(q^2\right)\left(\chi^8\left(q^2\right)-\chi^8\left(-q^2\right)\right),\end{aligned}$$

where we have used (36) in the right-hand side of the resulting identity. Further on replacing q by q^2 in (35) and then using in the above, we obtain

$$\chi^{16}\left(q^2\right)-\chi^8\left(-q^4\right)=\frac{16q^2\chi^8\left(q^2\right)}{\chi^8\left(-q^4\right)}.$$

Using (36) and (35) consecutively, we complete the proof. □

Theorem 13 *We have*

$$q(\psi^8(q) - \psi^8(-q)) = \varphi^8\left(q^2\right) - \varphi^8\left(-q^4\right).$$

Proof Using (13) in (8) and (9), we deduce

$$\psi^4(q) + \psi^4(-q) = 2\varphi^2\left(q^2\right)\psi^2\left(q^2\right) \tag{37}$$

and

$$\psi^4(q) - \psi^4(-q) = 8q\psi^8\left(q^2\right) \tag{38}$$

respectively. Now on squaring (13) and then replacing q by q^2, we obtain

$$\psi^2\left(q^4\right) = \frac{\psi^4\left(q^2\right)}{\varphi^2\left(q^2\right)}. \tag{39}$$

Multiplying (37) with (38), we obtain

$$\psi^8(q) - \psi^8(-q) = 16q\varphi^2\left(q^2\right)\psi^4\left(q^2\right)\psi^2\left(q^4\right).$$

Employing (39) in right-hand side of the above, we get

$$\psi^8(q) - \psi^8(-q) = 16q\psi^8\left(q^2\right). \tag{40}$$

On squaring (39), we obtain

$$\psi^8\left(q^2\right) = \varphi^4\left(q^2\right)\psi^4\left(q^4\right). \tag{41}$$

Replacing q by q^2 in (10) and (11), we see that

$$\varphi^4\left(q^2\right) - \varphi^4\left(-q^2\right) = 16q^2\psi^4\left(q^4\right) \tag{42}$$

and

$$\varphi\left(q^2\right)\varphi\left(-q^2\right) = \varphi^2\left(-q^4\right) \tag{43}$$

Using (41) in (40), we have

$$\begin{aligned}\psi^8(q) - \psi^8(-q) &= 16q\varphi^4\left(q^2\right)\psi^4\left(q^4\right)\\ &= \frac{1}{q}\varphi^4\left(q^2\right)\left\{\varphi^4\left(q^2\right) - \varphi^4\left(-q^2\right)\right\}\\ &= \frac{1}{q}\left\{\varphi^8\left(q^2\right) - \varphi^4\left(q^2\right)\varphi^4\left(-q^2\right)\right\},\end{aligned}$$

upon using (42) and (43). This completes the proof. □

Theorem 14 *We have*

$$3\varphi^3(q)\varphi^3\left(q^2\right) - 2\varphi(q)\varphi^5\left(q^2\right) - \varphi^5(q)\varphi\left(q^2\right) = 4qf^2(-q)f\left(-q^2\right)f\left(-q^4\right)f^2\left(-q^8\right).$$

Proof Consider

$$\begin{aligned}&3\varphi^3(q)\varphi^3\left(q^2\right) - 2\varphi(q)\varphi^5\left(q^2\right) - \varphi^5(q)\varphi\left(q^2\right)\\ &\quad= \varphi(q)\varphi\left(q^2\right)\left(3\varphi^2(q)\varphi^2\left(q^2\right) - 2\varphi^4\left(q^2\right) - \varphi^4(q)\right)\\ &\quad= \varphi(q)\varphi\left(q^2\right)\left(\varphi^2(q)\varphi^2\left(q^2\right) + 2\varphi^2(q)\varphi^2\left(q^2\right) - 2\varphi^4\left(q^2\right) - \varphi^4(q)\right)\end{aligned}$$

Using (8) and (11) in the right side of the second term, we deduce

$$\begin{aligned}&3\varphi^3(q)\varphi^3\left(q^2\right)-2\varphi(q)\varphi^5\left(q^2\right)-\varphi^5(q)\varphi\left(q^2\right)\\&\quad=\varphi(q)\varphi\left(q^2\right)\left\{\varphi^2(q)\varphi^2\left(q^2\right)+\varphi^4\left(-q^2\right)-2\varphi^4\left(q^2\right)\right\}\end{aligned}$$

On replacing q to q^2 in (10), and then employing in the above identity, we have

$$\begin{aligned}&3\varphi^3(q)\varphi^3\left(q^2\right)-2\varphi(q)\varphi^5\left(q^2\right)-\varphi^5(q)\varphi\left(q^2\right)\\&\quad=\varphi(q)\varphi\left(q^2\right)\left\{\varphi^2(q)\varphi^2\left(q^2\right)-16q^2\psi^4\left(q^4\right)-\varphi^4\left(q^2\right)\right\}\\&\quad=\varphi(q)\varphi\left(q^2\right)\left\{\varphi^2\left(q^2\right)\left(\varphi^2(q)-\varphi^2\left(q^2\right)\right)-16q^2\psi^4\left(q^4\right)\right\}\end{aligned}$$

On replacing q to $-q$ in (17) and employing in the above, we obtain

$$\begin{aligned}&3\varphi^3(q)\varphi^3\left(q^2\right)-2\varphi(q)\varphi^5\left(q^2\right)-\varphi^5(q)\varphi\left(q^2\right)\\&\quad=4q\varphi(q)\varphi\left(q^2\right)\psi^2\left(q^4\right)\left\{\varphi^2\left(q^2\right)-4q\psi^2\left(q^4\right)\right\}.\end{aligned}$$

Again using (17) and then (5) in the above, we obtain the desired result. □

3 Application to the theory of partition

The identities proved in Sect. 2 have applications in the theory of partitions. We demonstrate this by giving combinatorial interpretations for Theorem 1 and Theorem 11. For simplicity, we adopt the standard notation

$$(a_1, a_2, \ldots, a_n; q)_\infty := \prod_{j=1}^{n}(a_j; q)_\infty$$

and define

$$\left(q^{r\pm}; q^s\right)_\infty := \left(q^r, q^{s-r}; q^s\right)_\infty, \quad (r < s);\ r, s \in \mathbb{N}. \tag{44}$$

For example, $\left(q^{2\pm}; q^8\right)_\infty$ means $\left(q^2, q^6; q^8\right)_\infty$ which is $\left(q^2; q^8\right)_\infty\left(q^6; q^8\right)_\infty$.

Definition 1 A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called "colored partitions".

For example, if 1 is allowed to have 2 colors, then all the colored partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$ and $1_r + 1_g$. Where we use the indices r (red) and g (green) to distinguish the two colors of 1. Also

$$\frac{1}{\left(q^a; q^b\right)_\infty^k},$$

is the generating function for the number of partitions of n, where all the parts are congruent to $a \pmod{b}$ having k colors.

Theorem 15 *Let $P_1(n)$ denote the number of partitions of n into parts congruent to $+4 \pmod{8}$ with 4 colors. Let $P_2(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod{8}$ with 4 colors, parts congruent to $\pm 2 \pmod{8}$ with 6 colors and parts congruent to $\pm 3 \pmod{8}$ with 4 colors. Let $P_3(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 3 \pmod{8}$ with 4 colors, parts congruent to $\pm 2 \pmod{8}$ with 2 colors* *and parts congruent to* $+4 \pmod 8$ *with* 8 *colors. Then for any positive integer* $n \geq 1$*, the following equality holds:*

$$P_1(n) = P_2(n) - 4P_3(n-1).$$

Proof On employing (5) in Theorem 1, we obtain

$$\frac{1}{(q_2^2, q_{12}^4, q_2^6; q^8)_\infty} = \frac{1}{(q_4^1, q_8^2, q_4^3, q_8^4, q_4^5, q_8^6, q_4^7; q^8)_\infty} - \frac{4q}{(q_4^1, q_4^2, q_4^3, q_{16}^4, q_4^5, q_4^6, q_4^7; q^8)_\infty}$$

On canceling the common terms and then employing (44) in the above, we obtain

$$\frac{1}{\left(q_4^{4+}; q^8\right)_\infty} = \frac{1}{\left(q_4^{1\pm}, q_6^{2\pm}, q_4^{3\pm}; q^8\right)_\infty} - \frac{4q}{\left(q_4^{1\pm}, q_2^{2\pm}, q_4^{3\pm}, q_8^{4+}; q^8\right)_\infty}.$$

The three quotients of the above identity represents the generating function for $P_1(n)$, $P_2(n)$ and $P_3(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} P_1(n)q^n = \sum_{n=0}^{\infty} P_2(n)q^n - 4q \sum_{n=0}^{\infty} P_3(n)q^n,$$

where we set $P_1(0) = P_2(0) = P_3(0) = 1$. On equating the coefficients of q^n in the above, we lead to the desired result. □

The following table verifies the case when $n = 2$ in Theorem 15.

$P_1(2) = 0$	
$P_2(2) = 16$	$2_r, 2_w, 2_g, 2_b, 2_o, 2_y, 1_r + 1_r, 1_g + 1_g, 1_w + 1_w, 1_b + 1_b, 1_r + 1_b,$ $1_r + 1_w, 1_r + 1_b, 1_g + 1_w, 1_w + 1_b, 1_g + 1_b.$
$P_3(1) = 4$	$1_r, 1_g, 1_w, 1_b.$

Theorem 16 *Let* $P_1(n)$ *denote the number of partitions of n into parts congruent to* $\pm 2 \pmod 8$ *with* 10 *colors. Let* $P_2(n)$ *denote the number of partitions of n into parts congruent to* $\pm 2 \pmod 8$ *with* 6 *colors and parts congruent to* $+4 \pmod 8$ *with* 8 *colors and* $P_3(n)$ *denote the number of partitions of n into parts congruent to* $\pm 1, \pm 3, +4 \pmod 8$ *with* 4 *colors. Then for any positive integer* $n \geq 1$*, the following equality holds:*

$$P_1(n) + 4P_2(n-1) = P_3(n).$$

Proof Theorem 11 is equivalent to

$$\frac{1}{\left(q_{10}^{2\pm}; q^8\right)_\infty} + \frac{4q}{\left(q_6^{2\pm}, q_8^{4+}; q^8\right)_\infty} = \frac{1}{\left(q_4^{1\pm}, q_4^{3\pm}, q_4^{4+}; q^8\right)_\infty}.$$

Observe that, both left and right side of the above represents the generating function for $P_1(n)$, $P_2(n)$ and $P_3(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} P_1(n)q^n + 4q \sum_{n=0}^{\infty} P_2(n)q^n = \sum_{n=0}^{\infty} P_3(n)q^n,$$

where we set $P_1(0) = P_2(0) = P_3(0) = 1$. On equating the coefficients of q^n in the above, we lead to the desired result. □

The following table verifies the case when $n = 2$ in Theorem 16.

$P_1(2) = 10$	$2_r, 2_w, 2_g, 2_r, 2_p, 2_b, 2_y, 2_i, 2_{br}, 2_{bl}.$
$P_2(1) = 0$	
$P_3(2) = 10$	$1_r + 1_r, 1_b + 1_b, 1_w + 1_w, 1_g + 1_g, 1_r + 1_b, 1_r + 1_w, 1_r + 1_g, 1_b + 1_w, 1_w + 1_g, 1_b + 1_g.$

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