



Another refinement of the Pólya–Szegő inequality

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ARTICLE INFO

Article history:

Received 29 September 2009

Accepted 19 May 2010

Keywords:

Pólya–Szegő inequality in a triangle

Best constant

Chen's theorems

Sylvester's resultant

Discriminant sequence

Nonlinear algebraic equation systems

Computer softwares *Bottema* and *Maple*

(Version 9.0)

Pólya–Szegő inequality in a tetrahedron

Weitzenböck's inequality

ABSTRACT

In this paper, the authors make use of certain analytical techniques for nonlinear algebraic equation systems in order to give another refinement of the Pólya–Szegő inequality in a triangle, which is associated with one of Chen's theorems (see Chen (1993) [12] and Chen (2000) [13]). Some remarks and observations, as well as two closely-related open problems, are also presented.

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1. Introduction and the main results

For a given triangle ABC , we denote by a, b, c its side-lengths, by S its area, by p its semi-perimeter, and by R and r its circumradius and inradius, respectively.

In the year 1925, Georg Pólya (1887–1985) and Gábor Szegő (1895–1985) ([1, p. 161, Problem 17.1]; see also [2, p. 116]) proved the following beautiful and famous inequality which is known as the *Pólya–Szegő inequality* in the triangle ABC :

$$S \leq \frac{\sqrt{3}}{4} (abc)^{\frac{2}{3}}, \quad (1.1)$$

which may be compared with Weitzenböck's inequality in the triangle ABC (see, for example, [3, p. 42, Theorem 4.4]; see also [4, p. 112, Section 6.3]):

$$S \leq \frac{a^2 + b^2 + c^2}{4\sqrt{3}}$$

as well as another known inequality [3, p. 43, Theorem 4.5]:

$$S \leq \frac{ab + bc + ca}{4\sqrt{3}}.$$

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From among several extensions and modifications of the Pólya–Szegő inequality (1.1), we first recall the following sharpened version given by Leng [5] (see also [6, p. 194]):

$$S \leq \frac{\sqrt{3}}{4} (abc)^{\frac{2}{3}} \left(1 - \frac{(a-b)^2(b-c)^2(c-a)^2}{(abc)^2} \right)^{\frac{1}{6}}. \quad (1.2)$$

Chen [7] (see also [6,8]), on the other hand, strengthened the Pólya–Szegő's inequality (1.1) as follows:

$$S \leq \frac{\sqrt{3}}{4} (abc)^{\frac{2}{3}} \left(\frac{2r}{R} \right)^{\frac{1}{3}}. \quad (1.3)$$

More recently, Chen [9] gave a beautifully refined version of the Pólya–Szegő inequality (1.1), which we state here as [Theorem 1](#) below.

Theorem 1. *The best positive constant k for the following inequality:*

$$(abc)^{\frac{2}{3}} - \frac{4}{3}\sqrt{3}S \geq k \left(\frac{r}{R} \right) [(b+c-2a)^2 + (c+a-2b)^2 + (a+b-2c)^2] \quad (1.4)$$

is given by

$$k = F(x_0) \approx 0.12512379476902 \dots,$$

where

$$F(x) := \frac{(x+2)^2}{12x^2(x+1)} \left([4(x+2)^4]^{\frac{1}{3}} - \frac{4}{3}\sqrt{3}[(x+1)(x+3)]^{\frac{1}{2}} \right) \quad (x > 0)$$

and x_0 is one real root of the following equation:

$$6912(x+1)^3(5x^2+18x+12)^6 - (x+2)^8(x+3)^3(x^2-14x-12)^6 = 0.$$

The main object of this paper is to present yet another refinement of the Pólya–Szegő inequality (1.1) given by [Theorem 2](#) below.

Theorem 2. *The best positive constant k for the following inequality:*

$$\frac{\sqrt{3}}{4} (abc)^{\frac{2}{3}} - S \geq kr(R-2r) \quad (1.5)$$

is the real root on the interval $(1, \frac{23}{20})$ of the following equation:

$$\begin{aligned} &80621568k^{26} - 1169012736k^{24} + 2306112768k^{22} - 1986308842752k^{20} \\ &- 271161740638512k^{18} - 7075252951678008k^{16} - 72860319298449837k^{14} \\ &- 315039331520882532k^{12} + 143128010909935188k^{10} + 407040335182644176k^8 \\ &+ 175081049919823564k^6 - 18908198108992k^4 + 539361792k^2 - 5184 = 0. \end{aligned} \quad (1.6)$$

Furthermore, the constant k has its numerical approximation given by

$$k \approx 1.145209656 \dots$$

2. Preliminary results and lemmas

In order to prove [Theorem 2](#), we require several lemmas.

Lemma 1. *If the following inequality:*

$$\frac{\sqrt{3}}{4} (abc)^{\frac{2}{3}} - S \geq kr(R-2r) \quad (k > 0) \quad (2.1)$$

holds true, then

$$0 < k \leq \frac{3}{4}\sqrt{3}.$$

Proof. First of all, Chen [7] (see also [8]) derived the following inequality:

$$\frac{\sqrt{3}}{4} (abc)^{\frac{2}{3}} \leq S \left(\frac{R}{2r} \right)^{\frac{1}{2}}. \quad (2.2)$$

By using Chen’s inequality (2.2), we find that

$$S \left(\frac{R}{2r} \right)^{\frac{1}{2}} - S \geq \frac{\sqrt{3}}{4} (abc)^{\frac{2}{3}} - S \geq kr \quad (R - 2r).$$

Thus, in view of the known identity $S = rp$, we get the following inequality:

$$k \leq \frac{\left(\sqrt{\frac{R}{2r}} - 1 \right) p}{R - 2r} = \frac{p}{\sqrt{2Rr} + 2r}.$$

Consequently, we have

$$k \leq \min \left(\frac{p}{\sqrt{2Rr} + 2r} \right).$$

By means of the following known inequalities [3, p. 52]:

$$p^2 \geq \frac{27}{2} Rr \quad \text{and} \quad p \geq 3\sqrt{3}r,$$

we obtain

$$\frac{p}{\sqrt{2Rr} + 2r} \geq \frac{p}{\sqrt{\frac{4p^2}{27} + \frac{2p}{3\sqrt{3}}}} = \frac{3}{4} \sqrt{3}. \tag{2.3}$$

The inequality (2.3) holds true if and only if the triangle is an equilateral triangle. So

$$\min \left(\frac{p}{\sqrt{2Rr} + 2r} \right) = \frac{3\sqrt{3}}{4}.$$

We then find that

$$k \leq \frac{3}{4} \sqrt{3}.$$

Our proof of Lemma 1 is thus completed. \square

Lemma 2 (See [10,11]). For a polynomial $p(x)$ with real coefficients given by

$$p(x) := a_0x^n + a_1x^{n-1} + \dots + a_n,$$

if the number of the sign changes of the revised sign list of its discriminant sequence:

$$\{D_1(p), D_2(p), \dots, D_n(p)\}$$

is v , then the number of the pairs of distinct conjugate imaginary roots of $p(x)$ equals v . Furthermore, if the number of non-vanishing members of the revised sign list is l , then the number of the distinct real roots of $p(x)$ equals $l - 2v$.

Lemma 3 (See [12–14]). Let $G(R, r, p)$ be a function of the measurements R, r and p for a triangle. Suppose also that the functions $f_1(R, r)$ and $f_2(R, r)$ depend upon R and r .

(i) If the following homogeneous inequality in a triangle:

$$G(R, r, p) \geq 0 \quad (> 0), \tag{2.4}$$

which is equivalent to the inequality:

$$p \geq (>)f_1(R, r),$$

holds true for any isosceles triangle whose top angle is greater than or equal to 60° , then the inequality (2.4) holds true for any triangle.

(ii) If the homogeneous inequality (2.4) in a triangle, which is equivalent to the following inequality:

$$p \leq (<)f_2(R, r)$$

holds true for any isosceles triangle whose top angle is less than or equal to 60° , then the inequality (2.4) holds true for any triangle.

Lemma 4 (See [11]). Define the polynomials $f(x)$ and $g(x)$ by

$$f(x) := a_0x^n + a_1x^{n-1} + \dots + a_n$$

and

$$g(x) := b_0x^m + b_1x^{m-1} + \dots + b_m.$$

If

$$a_0 \neq 0 \text{ or } b_0 \neq 0,$$

then the polynomials $f(x)$ and $g(x)$ have common roots if and only if

$$R(f, g) = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-1} & a_n & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_0 & \dots & \dots & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & \dots & \dots & \dots & 0 \\ 0 & b_0 & b_1 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_0 & b_1 & \dots & b_m \end{vmatrix} = 0,$$

where $R(f, g)$ is Sylvester's resultant of $f(x)$ and $g(x)$.

3. Demonstration of Theorem 2

In this section, we apply the results and lemmas of the preceding section in order to prove Theorem 2.

Proof. In light of the known identities [15, p. 52]:

$$abc = 4Rrp \text{ and } S = rp,$$

the inequality (1.5) is equivalent to the following inequality:

$$\frac{\sqrt{3}}{4}(4Rrp)^{\frac{2}{3}} - rp \geq kr(R - 2r). \tag{3.1}$$

Furthermore, the inequality (3.1) is equivalent to the following inequality:

$$\frac{3}{4}\sqrt{3}R^2r^2p^2 \geq r^3[p + k(R - 2r)]^3. \tag{3.2}$$

Consequently, we have

$$rp^3 + \left(3k(R - 2r)r - \frac{3}{4}\sqrt{3}R^2\right)p^2 + 3k^2(R - 2r)^2rp + k^3(R - 2r)^3r \leq 0. \tag{3.3}$$

Obviously, this last inequality (3.3) holds true when $R = 2r$. In the case when $R > 2r$, we define a polynomial $h(p)$ by

$$h(p) := rp^3 + \left(3k(R - 2r)r - \frac{3}{4}\sqrt{3}R^2\right)p^2 + 3k^2(R - 2r)^2rp + k^3(R - 2r)^3r.$$

Then the discriminant sequence of $h(t^2)$ is given by

$$\left[r^2, \varphi_1(R, r) \cdot r^3, \varphi_1(R, r) \cdot \varphi_2(R, r) \cdot R^2r^3, k^2 \cdot \varphi_2(R, r) \cdot \varphi_3(R, r) \cdot (R - 2r)^2R^4r^4, k^5 \cdot \varphi_3(R, r) \cdot \varphi_4(R, r) \cdot (R - 2r)^5R^6r^5, -k^9 \cdot \varphi_4^2(R, r) \cdot (R - 2r)^9R^8r^6\right],$$

where

$$\begin{aligned} \varphi_1(R, r) &:= 8kr^2 - 4rkR + \sqrt{3}R^2, \\ \varphi_2(R, r) &:= 16kr^2 - 8rkR + \sqrt{3}R^2, \\ \varphi_3(R, r) &:= 3\sqrt{3}R^2 - 28rkR + 56kr^2 \end{aligned}$$

and

$$\varphi_4(R, r) := 18kr^2 - 9rkR + \sqrt{3}R^2.$$

By applying Lemma 1 and the fact that $R > 2r$, the following four inequalities:

$$\varphi_i(R, r) > 0 \quad (i = 1, 2, 3, 4)$$

hold true obviously. Then the revised sign list of the discriminant sequence of $h(t^2)$ is just as given below:

$$[1, 1, 1, 1, 1, -1]. \tag{3.4}$$

The number of the sign changes of (3.4) is 1. Thus, in view of Lemma 2, the polynomial $h(t^2)$ has 4 distinct real roots. Moreover, the polynomial $h(p)$ has 2 distinct positive real roots (see, for details, [16]). So the inequality (3.3) can be rewritten in its equivalent form:

$$f_1(R, r) \leq p \leq f_2(R, r).$$

By making use of Lemma 3, we easily see that the inequality (3.3) holds true if and only if the triangle is an isosceles triangle. We now let

$$a = 2 \quad \text{and} \quad b = c = x \quad (x > 1).$$

Then the inequality (1.5) is equivalent to the following inequality:

$$\frac{\sqrt{3}}{4} (2x^2)^{\frac{2}{3}} - \sqrt{x^2 - 1} \geq k \left(\frac{(x - 2)^2}{2(x + 1)} \right). \tag{3.5}$$

- (i) In the case when $x = 2$, the inequality (3.5) holds true obviously.
- (ii) In the case when

$$x > 1 \quad \text{and} \quad x \neq 2,$$

the inequality (3.5) is seen to be equivalent to the following inequality:

$$k \leq \frac{(x + 1) \left(\sqrt{3}(4x^4)^{\frac{1}{3}} - 4\sqrt{x^2 - 1} \right)}{2(x - 2)^2}. \tag{3.6}$$

Define the function $H(x)$ by

$$H(x) := \frac{(x + 1) \left(\sqrt{3}(4x^4)^{\frac{1}{3}} - 4\sqrt{x^2 - 1} \right)}{2(x - 2)^2} \quad (x \in (1, 2) \cup (2, \infty)).$$

By calculating the derivative for $H(x)$, we get

$$H'(x) = \frac{\sqrt{3}^{\frac{3}{4}} 4^{\frac{3}{4}} \sqrt{x} \sqrt{(x - 1)(x + 1)} (x^2 - 16x - 8) + 12(x + 1)(5x - 4)}{6(x - 2)^3 \sqrt{x^2 - 1}},$$

which, upon setting $H'(x) = 0$, yields

$$\sqrt{3}^{\frac{3}{4}} 4^{\frac{3}{4}} \sqrt{x} \sqrt{(x - 1)(x + 1)} (x^2 - 16x - 8) + 12(x + 1)(5x - 4) = 0. \tag{3.7}$$

It is easy to find from (3.7) that

$$x^2 - 16x - 8 < 0,$$

which implies that

$$1 < x < 2 \quad \text{or} \quad 2 < x < 8 + 6\sqrt{2}.$$

It is not difficult to observe that the roots of Eq. (3.7) must be the same as the roots of the following equation:

$$(x^{13} - 95x^{12} + 3703x^{11} - 74949x^{10} + 808572x^9 - 4034688x^8 + 3454464x^7 + 13215792x^6 - 15891072x^5 - 11578112x^4 + 17747968x^3 + 790528x^2 - 6193152x + 1769472)(x + 2)(x + 1)^3(x - 2)^3 = 0. \tag{3.8}$$

Since the range of the roots of Eq. (3.7) is given by

$$(1, 2) \cup (2, 8 + 6\sqrt{2}),$$

the roots of Eq. (3.7) must be the same as the roots of the following equation:

$$x^{13} - 95x^{12} + 3703x^{11} - 74949x^{10} + 808572x^9 - 4034688x^8 + 3454464x^7 + 13215792x^6 - 15891072x^5 - 11578112x^4 + 17747968x^3 + 790528x^2 - 6193152x + 1769472 = 0. \tag{3.9}$$

Now, if we define the polynomial $q(x)$ by

$$q(x) := x^{13} - 95x^{12} + 3703x^{11} - 74949x^{10} + 808572x^9 - 4034688x^8 + 3454464x^7 + 13215792x^6 - 15891072x^5 - 11578112x^4 + 17747968x^3 + 790528x^2 - 6193152x + 1769472, \tag{3.10}$$

then the revised sign list of the discriminant sequence of $q(x)$ is given as follows:

$$[1, 1, -1, 1, 1, 1, 1, 1, 1, -1, 1, -1]. \quad (3.11)$$

Therefore, in view of Lemma 2, we know that Eq. (3.9) has 5 pairs of distinct conjugate imaginary roots and 3 distinct real roots. For

$$q(-2) < 0, \quad q(0) > 0, \quad q(2) > 0, \quad q(3) < 0, \quad q(17) < 0 \quad \text{and} \quad q(24) > 0,$$

we know that Eq. (3.9) has only one real root on the interval $(1, 2) \cup (2, 8 + 6\sqrt{2})$.

Denote by

$$x_0 = 2.337099889 \dots$$

the root of Eq. (3.9) which lies in the interval $(2, 3)$. Then

$$\begin{aligned} \min\{H(x)\} &=: H(x_0) = \frac{(x_0 + 1) \left(\sqrt{3}(4x_0^4)^{\frac{1}{3}} - 4\sqrt{x_0^2 - 1} \right)}{2(x_0 - 2)^2} \\ &= 1.1452096 \dots \in \left(1, \frac{23}{20} \right). \end{aligned} \quad (3.12)$$

It, therefore, follows that the maximum value of k is $H(x_0)$.

We next prove that $H(x_0)$ is the root of Eq. (1.6). For this purpose, we consider the following nonlinear algebraic equation system:

$$\begin{cases} (x_0 + 1)(u_0 - v_0) - 2(x_0 - 2)^2 t = 0 \\ u_0^6 - 432x_0^8 = 0 \\ v_0^2 - 16x_0^2 + 16 = 0 \\ h(x_0) = 0. \end{cases} \quad (3.13)$$

It is easy to see that $H(x_0)$ is also the solution of the nonlinear algebraic equation system (3.13). If we eliminate the u_0 , v_0 and x_0 ordinals by resultant (by using Lemma 4), then we get

$$p_1(t)p_2(t)p_3(t)p_4(t) = 0, \quad (3.14)$$

where

$$\begin{aligned} p_1(t) &:= 1289945088 t^{26} - 80152672960512 t^{24} - 112148121563136 t^{22} - 61391248256544768 t^{20} \\ &\quad - 2341074066668464896 t^{18} - 7182680904477244800 t^{16} + 153376610542407735984 t^{14} \\ &\quad - 118924209115815414240 t^{12} - 701301826334736491400 t^{10} + 3562415035017469718768 t^8 \\ &\quad + 10364657150848707001675 t^6 - 55040931733349010016 t^4 \\ &\quad + 31697987689208832 t^2 - 7549987180176, \\ p_2(t) &:= 80621568 t^{26} - 1169012736 t^{24} + 2306112768 t^{22} - 1986308842752 t^{20} - 271161740638512 t^{18} \\ &\quad - 7075252951678008 t^{16} - 72860319298449837 t^{14} - 315039331520882532 t^{12} \\ &\quad + 143128010909935188 t^{10} + 407040335182644176 t^8 \\ &\quad + 175081049919823564 t^6 - 18908198108992 t^4 + 539361792 t^2 - 5184, \\ p_3(t) &:= 109049173118505959030784 t^{52} \\ &\quad + 1731646344535315376429334528 t^{50} \\ &\quad + 26176397897252997561239564451840 t^{48} \\ &\quad - 2157113478743472316050391009591296 t^{46} \\ &\quad + 96285828119422129204570177941798912 t^{44} \\ &\quad - 4620237125675523872910410325034008576 t^{42} \\ &\quad + 18310474736736595599527123831733878784 t^{40} \\ &\quad + 1660928888587650856918069069179254734848 t^{38} \\ &\quad - 31593208873012526030049394182328317640704 t^{36} \\ &\quad + 4690026916103512850737032526548866722430976 t^{34} \\ &\quad + 134813681107769609317603243217463634984697856 t^{32} \\ &\quad + 734979188543992577890673868750641219672997888 t^{30} \end{aligned}$$

$$\begin{aligned}
 & - 5053568929290494598689554518897577791776686080 t^{28} \\
 & - 43391037892440148424133203652063242439686750208 t^{26} \\
 & + 556133885780667323878061294504685472263264878592 t^{24} \\
 & - 1262878804397138107015109951458494178122338107392 t^{22} \\
 & - 1397118451774316048319200483729727853340468062464 t^{20} \\
 & + 5844608792227356420167656517180609331523722701568 t^{18} \\
 & + 14506317033659270608132528797103125132048309916576 t^{16} \\
 & + 15881156573368175727834239369823051173835954591376 t^{14} \\
 & + 7251649956998284021580488430996425029404359926489 t^{12} \\
 & + 16890973769944612500834739229379176877309052576 t^{10} \\
 & + 16326232453597840365093106690850683154726656 t^8 \\
 & + 4697513294781725869890298817642188956576 t^6 \\
 & + 661302759480584975633209470927450624 t^4 \\
 & + 50195901378320989663410067857408 t^2 \\
 & + 1787630646346469333194109184
 \end{aligned}$$

and

$$\begin{aligned}
 p_4(t) := & 109049173118505959030784 t^{52} \\
 & - 5042706387932511810481029120 t^{50} \\
 & + 237185531752279478954071525687296 t^{48} \\
 & - 16014880610674766168713970621349888 t^{46} \\
 & + 172305658761861817856339903325929472 t^{44} \\
 & - 1374341528145008523229874965287272448 t^{42} \\
 & - 363944398148296975427328378161174937600 t^{40} \\
 & + 17870285068446288294573103221072394715136 t^{38} \\
 & + 572016176152015467009339313219645068017664 t^{36} \\
 & + 1976057760202715502842240634133562165035008 t^{34} \\
 & - 23780408261875252593745102833056765190340608 t^{32} \\
 & + 279284379799342130859337716413966139119370240 t^{30} \\
 & + 2868165962302296804653584111904006054064095232 t^{28} \\
 & - 17926414041425242615175719877591591415146741760 t^{26} \\
 & - 86540271722115435563245881995144794203767881728 t^{24} \\
 & + 91913668928116760451784769156036687629772980224 t^{22} \\
 & + 1442318336309549831979159245819747967508523923200 t^{20} \\
 & + 4653225657821428741228779880161897056765597252352 t^{18} \\
 & + 7854535278157154395904067458515096323570177067936 t^{16} \\
 & + 7657808127260536519462564306169403708733858520080 t^{14} \\
 & + 3681313578029876351022920680459739569535303339385 t^{12} \\
 & - 10481487670276498331083512816840708439929120992 t^{10} \\
 & + 48536765456337201047605841282542168844742400 t^8 \\
 & - 10990122608200382363389997436230901430368 t^6 \\
 & + 3435397925462976127637720888829762048 t^4 \\
 & - 1212516161444277820257536568434688 t^2 \\
 & + 376130813743642388301942591744.
 \end{aligned}$$

The revised sign list of the discriminant sequence of $p_1(t)$ is given by

$$[1, 1, 1, -1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, -1, -1, 1, 1, 1, 1, 1, 1, -1, 1]. \tag{3.15}$$

The revised sign list of the discriminant sequence of $p_3(t)$ is given by

$$[1, -1, 1, 1, 1, 1, -1, -1, -1, 1, 1, -1, -1, 1, 1, 1, -1, 1, 1, 1, 1, -1, -1, 1,$$

$$-1, 1, 1, 1, -1, 1, -1, -1, -1, 1, -1, -1, -1, -1, 1, -1, -1, -1, 1, -1, 1, 1, 1, -1, 1, 1]. \tag{3.16}$$

The revised sign list of the discriminant sequence of $p_4(t)$ is given by

$$[1, 1, -1, 1, 1, 1, 1, 1, -1, 1, -1, -1, 1, -1, -1, -1, 1, -1, 1, 1, 1, -1, -1, -1, 1, -1, -1, 1, -1, 1, -1, -1, 1, 1, 1, -1, 1, 1, 1, -1, 1]. \tag{3.17}$$

So the number of the sign changes of the revised sign list of (3.15) is 10. Thus, by applying Lemma 2, we find that the following equation:

$$p_1(t) = 0 \tag{3.18}$$

has 6 distinct real roots. Also, by using the function “realroot(· · ·)” in Maple (Version 9.0) [17, pp. 110–114], we can find that Eq. (3.18) has 6 distinct real roots in the following intervals:

$$\left[\frac{1}{16}, \frac{1}{8}\right], \left[\frac{19}{8}, \frac{39}{16}\right], \left[\frac{997}{4}, \frac{3989}{16}\right], \left[-\frac{1}{8}, -\frac{1}{16}\right], \left[-\frac{39}{16}, -\frac{19}{8}\right] \text{ and } \left[-\frac{3989}{16}, -\frac{997}{4}\right]. \tag{3.19}$$

So Eq. (3.18) has no real root on the interval $(1, \frac{23}{20})$. Moreover, the number of the sign changes of the revised sign list of (3.16) and (3.17) are both 26. Thus, by appealing to Lemma 2, we see that the following equations:

$$p_3(t) = 0 \tag{3.20}$$

and

$$p_4(t) = 0 \tag{3.21}$$

have both 26 pairs of distinct conjugate imaginary roots. Consequently, Eqs. (3.20) and (3.21) have no real root.

From (3.12), we can find that $H(x_0)$ is the root of the following equation:

$$p_2(t) = 0. \tag{3.22}$$

It follows that $H(x_0)$ is the root of Eq. (1.6).

The proof of Theorem 2 is thus completed. □

4. Remarks and observations

In this section, we present a number of remarks and observations which are relevant to the foregoing developments.

Remark 1. By applying the above analytical techniques *mutatis mutandis*, we can also show that the best positive constant k for the inequality (1.4) is the real root on the interval $(\frac{1}{11}, \frac{1}{7})$ of the following equation:

$$\begin{aligned} & 1711559752519106944 k^{19} + 316248778897380864 k^{18} - 3800109748278481632 k^{17} \\ & - 11531837192336629407 k^{16} + 2607254040139319556 k^{15} \\ & - 56760406902842186385 k^{14} + 3751820005736319930 k^{13} \\ & + 8268108002201410434 k^{12} + 12069294416915771034 k^{11} \\ & - 1042069673906565390 k^{10} + 2878227242413204194 k^9 \\ & - 666644248788536628 k^8 + 47871914625009990 k^7 \\ & - 1369374355945116 k^6 + 5003949589506 k^5 - 102324963501 k^4 \\ & - 278510508 k^3 - 3222288 k^2 + 576 k - 32 = 0. \end{aligned} \tag{4.1}$$

Remark 2. By means of the software *Bottema* (see [18–20]) which was invented by Lu Yang, we cannot only obtain the same result as above, but also find that the best positive constant k for the following inequality:

$$\frac{\sqrt{3}}{4} (abc)^{\frac{2}{3}} - S \leq kR(R - 2r) \tag{4.2}$$

is the real root on the interval $(\frac{11}{19}, \frac{7}{12})$ of the equation given below:

$$171382426877952 k^{44} - 18337919675940864 k^{42} + 911698970389401600 k^{40} - 27829451391907737600 k^{38} + 582228104028327869184 k^{36}$$

$$\begin{aligned}
 & - 8731603453259044776960 k^{34} + 95562077894969708786688 k^{32} \\
 & - 782962684661804314014720 k^{30} + 4977711054259411004720640 k^{28} \\
 & - 23445637706440079778719520 k^{26} + 77857860174501407407054848 k^{24} \\
 & - 85235528600240500366155072 k^{22} + 46539122023111810136151600 k^{20} \\
 & - 15072105390562765604035152 k^{18} + 1024477253725366529541891 k^{16} \\
 & + 128697762399197003007048 k^{14} + 102562175760518598173748 k^{12} \\
 & - 6838928862950990759368 k^{10} + 163979025717344216850 k^8 \\
 & - 1492951540781714760 k^6 + 1918013888842308 k^4 \\
 & - 856636125192 k^2 + 129140163 = 0.
 \end{aligned} \tag{4.3}$$

Moreover, the constant k can be numerically approximated by

$$k \approx 0.5800733927 \dots$$

Remark 3. We perform all of the aforementioned operations in this paper with the computer software *Maple* (Version 9.0).

5. A set of open problems

In this concluding section of our paper, we pose two closely-related problems which would refine the Pólya–Szegő inequality in a tetrahedron (see [21, pp. 188 and 197]).

Problem 1. Let S_k ($k = 1, 2, 3, 4$) denote the area of the face of a given tetrahedron and let V be the volume of the tetrahedron. Suppose also that R and r are the circumradius and the inradius of the tetrahedron, respectively. Determine the best constants K_1 and K_2 for the following two inequalities:

$$\frac{2^{\frac{3}{2}}}{3^{\frac{7}{4}}} \left(\prod_{k=1}^4 S_k \right)^{\frac{3}{8}} - V \geq K_1 r^2 (R - 3r) \tag{5.1}$$

and

$$\frac{2^{\frac{3}{2}}}{3^{\frac{7}{4}}} \left(\prod_{k=1}^4 S_k \right)^{\frac{3}{8}} - V \geq K_2 Rr (R - 3r). \tag{5.2}$$

Problem 2. Let

$$\rho_{ij} = |A_i A_j| \quad (i, j = 1, 2, 3, 4; i \neq j)$$

denote the length of the edge of a given tetrahedron and let V be the volume of the tetrahedron. Suppose also that R and r are the circumradius and the inradius of the tetrahedron, respectively. Determine the best constants \mathfrak{K}_1 and \mathfrak{K}_2 for the following two inequalities:

$$\frac{\sqrt{2}}{12} \left(\prod_{1 \leq i < j \leq 4} \rho_{ij}^{\frac{1}{2}} \right) - V \geq \mathfrak{K}_1 r^2 (R - 3r) \tag{5.3}$$

and

$$\frac{\sqrt{2}}{12} \left(\prod_{1 \leq i < j \leq 4} \rho_{ij}^{\frac{1}{2}} \right) - V \geq \mathfrak{K}_2 Rr (R - 3r). \tag{5.4}$$

Each of these two *Open Problems* has challenged the authors for quite sometime. The solutions to either or both of these problems (if and when found by any interested reader) would naturally interest the authors, too, a great deal.

References

[1] G. Pólya, G. Szegő, Problems and Theorems in Analysis, Vol. II, in: Grundlehren der Mathematischen Wissenschaften, Band 20, Springer-Verlag, New York, Heidelberg and Berlin, 1976, (Translated from the revised and enlarged Fourth German Edition by C. E. Billigheimer).
 [2] J. Sándor, On the geometry of equilateral triangles, Forum Geom. 5 (2005) 107–117.
 [3] O. Bottema, R.Ž. Djorđević, R.R. Janić, D.S. Mitrinović, P.M. Vasić, Geometric Inequalities, Wolters-Noordhoff Publishing Company, Groningen, 1969.

- [4] D.S. Mitrinović, E.S. Barnes, D.C.B. Marsh, J.R.M. Radok, *Elementary Inequalities*, P. Noordhoff Limited, Groningen, 1964.
- [5] G.-S. Leng, Two inequalities in tetrahedron, *Hunan Math. Comm.* 2 (1986) 11–13 (in Chinese).
- [6] J.-C. Kuang, *Applied Inequalities*, 3rd edition, Shangdong Science and Technology Press, Ji'nan, 2003 (in Chinese).
- [7] S.-L. Chen, An inequality chain relating to several famous inequalities, *Hunan Math. Comm.* 1 (1995) 41 (in Chinese).
- [8] X.-Z. Yang, *Research in Inequalities*, No. 601–602, Tibet People's Press, Lhasa, 2000 (in Chinese).
- [9] S.-L. Chen, A sharpening on Pólya–Szegő's inequalities, *High-School Math.* 12 (2005) 36–37 (in Chinese).
- [10] L. Yang, X.-R. Hou, Z.-B. Zeng, A complete discrimination system for polynomials, *Sci. China Ser. E* 39 (1996) 628–646.
- [11] L. Yang, J.-Z. Zhang, X.-R. Hou, *Nonlinear Algebraic Equation System and Automated Theorem Proving*, No. 23–25, Shanghai Scientific and Technological Education Press, Shanghai, 1996 (in Chinese).
- [12] S.-L. Chen, A new method to prove one kind of inequalities: equate substitution method, *Fujian High-School Math.* 3 (1993) 20–23 (in Chinese).
- [13] S.-L. Chen, The simplified method to prove inequalities in triangle, in: *Research in Inequalities*, No. 3–8, Tibet People's Press, Lhasa, 2000 (in Chinese).
- [14] Y.-D. Wu, The best constant for a geometric inequality, *J. Inequal. Pure Appl. Math.* 6 (4) (2005) 1–8. Article 111 (electronic).
- [15] D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, Boston and London, 1989.
- [16] B.-C. Xia, L. Yang, Some properties of the discrimination matrix of polynomials with applications, *Acta Math. Appl. Sinica* 26 (2003) 652–663 (in Chinese).
- [17] D.-M. Wang, *Selected Lectures in Symbolic Computation*, Tsinghua University Press, Beijing, 2003 (in Chinese).
- [18] L. Yang, A dimension-decreasing algorithm with generic program for automated inequality proving, *Chinese High Tech. Lett.* 7 (1998) 20–25 (in Chinese).
- [19] L. Yang, The symbolic algorithm for global optimization with finiteness principle, in: *Mathematics and Mathematics-Mechanization*, Shandong Education Press, Ji'nan, 2001, pp. 210–220 (in Chinese).
- [20] L. Yang, S.-H. Xia, Automated proving for a class of constructive geometric inequalities, *Chinese J. Comput.* 26 (2003) 769–778 (in Chinese).
- [21] W.-X. Shen, *Introduction to Simplicies*, Hunan Normal University Press, Changsha, 2000 (in Chinese).