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A Note on Minimal Dominating Signed Graphs

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Abstract: In this note, we study minimal dominating signed graphs and obtain structural characterization of minimal dominating signed graphs. Further, we characterize signed graphs S for which $MD(S) \sim CMD(S)$, where $\sim$ denotes switching equivalence and $MD(S)$ and $CMD(S)$ are denotes the minimal dominating signed graph and common minimal dominating signed graph of S respectively.

Key Words: Signed graphs, balance, switching, complement, minimal dominating signed graphs, common minimal signed graphs, negation.

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§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [5]; the non-standard will be given in this paper when required. We treat only finite simple graphs without self loops and isolates.

A signed graph is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$, called the underlying graph of S and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$, called the signature (or sign in short) of S. Alternatively, the signed graph can be written as $S = (V, E, \sigma)$, with V, E, σ in the above sense. Let $E^+(S) = \{e \in E : \sigma(e) = +\}$ and $E^-(S) = \{e \in E : \sigma(e) = -\}$. The elements of $E^+(S)$ and $E^-(S)$ are called positive and negative edges of S, respectively. A signed graph is all-positive (respectively, all-negative) if all its edges are positive (negative).

A cycle in a signed graph S is said to be *positive* if it contains an even number of negative edges. A given signed graph S is said to be *balanced* if every cycle in S is positive (see [6]). In a signed graph $S = (S^u, \sigma)$, for any $A \subseteq E$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A. For more new notions on signed graphs refer the papers ([11, 12, 15, 16], [18]-[24]).

A marked signed graph is an ordered pair $S_\mu = (S, \mu)$, where $S = (S^u, \sigma)$ is a signed graph and $\mu : V(S^u) \to \{+, -\}$ is a function from the vertex set $V(S^u)$ of S^u into the set $\{+, -\}$,

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called a *marking* of S. In particular, σ induces a unique marking μ_σ defined by

$$\mu_\sigma(v) = \prod_{e \in E_v} \sigma(e),$$

where E_v is the set of edges incident at v in S, is called the *canonical marking* of S. We shall denote by $\mathcal{M}_S$ the set of all markings of S. A signed graph S together with one of its markings μ is denoted by S_μ .

The following characterization of balanced signed graphs is well known.

Proposition 1.1(E.Sampathkumar [14]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Given a marking μ of S, by switching S with respect to μ we mean changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs in S_μ . The signed graph obtained in this way is denoted by $S_\mu(S)$ and is called the μ-switched signed graph or just switched signed graph when the marking is clear from the context (Sampthkumar et al. [17]).

We say that signed graph S_1 *switches to signed graph* S_2 (or that they are *switching equivalent* to each other), written as $S_1 \sim S_2$, whenever there exists $\mu \in \mathcal{M}_{S_1}$ such that $S_\mu(S_1) \cong S_2$, where "$\cong$" denotes the isomorphism between any two signed graphs in the standard sense. Note that $S_1 \sim S_2$ implies that $(S_1)^u \cong (S_2)^u$.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [26]) or *cycle isomorphic* (see [28]) if there exists an isomorphism $f : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $f(Z)$ in S_2. The following result will also be useful in our further investigation.

Proposition 1.2(T.Zaslavsky [28]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

In [17], the authors introduced the switching and cycle isomorphism for signed digraphs.

§2. Minimal Dominating Signed Graph

Mathematical study of domination in graphs began around 1960, there are some references to domination-related problems about 100 years prior. In 1862, de Jaenisch [3] attempted to determine the minimum number of queens required to cover an $n \times n$ chess board. In 1892, W. W. Rouse Ball [13] reported three basic types of problems that chess players studied during that time.

The study of domination in graphs was further developed in the late 1950s and 1960s, beginning with Berge [1] in 1958. Berge wrote a book on graph theory, in which he introduced the "coefficient of external stability", which is now known as the domination number of a graph. Oystein Ore [10] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962. The problems described above were studied in more detail around 1964 by brothers Yaglom and Yaglom [27]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops. A decade later, Cockayne and Hedetniemi [2] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D of G is minimal, if for any vertex $v \in D$, $D - \{v\}$ is not a dominating set of G (See, Ore [10]).

Let $\mathcal{S}$ be a finite set and $F = \{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n\}$ be a partition of S. Then the *intersection graph* $\Omega(F)$ of F is the graph whose vertices are the subsets in F and in which two vertices $\mathcal{S}_i$ and $\mathcal{S}_j$ are adjacent if and only if $\mathcal{S}_i \cap \mathcal{S}_j \neq \phi$, $i \neq j$.

Kulli and Janakiram [8] introduced a new class of intersection graphs in the field of domination theory. The *minimal dominating graph* $MD(G)$ of a graph G is the intersection graph defined on the family of all minimal dominating sets of vertices in G.

We now extend the notion of $MD(G)$ to the realm of signed graphs. The *minimal dominating signed graph* $MD(S)$ of a signed graph $S = (S^u, \sigma)$ is a signed graph whose underlying graph is $MD(G)$ and sign of any edge PQ in $MD(S)$ is $\mu(P)\mu(Q)$, where μ is the canonical marking of S, P and Q are any two minimal dominating sets of vertices in S^u. Further, a signed graph $S = (G, \sigma)$ is called minimal dominating signed graph, if $S \cong MD(S')$ for some signed graph S'. In this paper we will give a structural characterization of which signed graphs are common minimal dominating signed graph. The following result indicates the limitations of the notion $CMD(S)$ introduced above, since the entire class of unbalanced signed graphs is forbidden to be minimal dominating signed graphs.

Proposition 2.1 *For any signed graph $S = (G, \sigma)$, its minimal dominating signed graph $MD(S)$ is balanced.*

Proof Since sign of any edge PQ in $MD(S)$ is $\mu(P)\mu(Q)$, where μ is the canonical marking of S, by Proposition 1.1, $MD(S)$ is balanced. □

For any positive integer k, the k^{th} iterated minimal dominating signed graph $MD(S)$ of S is defined as follows:

$$MD^0(S) = S,\ MD^k(S) = MD(MD^{k-1}(S))$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and any positive integer k, $MD^k(S)$ is balanced.*

Proposition 2.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, their minimal dominating signed graphs are switching equivalent.*

Proof Suppose $S_1 = (S_1^u, \sigma)$ and $S_2 = (S_2^u, \sigma')$ be two signed graphs with $S_1^u \cong S_2^u$. By Proposition 2.1, $MD(S_1)$ and $MD(S_2)$ are balanced and hence, the result follows from Proposition 1.2. □

In [25], the authors introduced the notion common minimal dominating signed graph of a signed graph as follows:

A *common minimal dominating signed graph* $CMD(S)$ of a signed graph $S = (G, \sigma)$ is such a signed graph whose underlying graph is $CMD(G)$ and sign of any edge uv in $CMD(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S.

The following result restricts the class of minimal dominating graphs.

Proposition 2.4 *For any signed graph $S = (G, \sigma)$, its common minimal dominating signed graph $CMD(S)$ is balanced.*

We now characterize the signed graphs whose minimal dominating signed graphs and common minimal dominating signed graphs are switching equivalent. In case of graphs the following result is due to Kulli and Janakiram [9]:

Proposition 2.5(Kulli and Janakiram [9]) *If G is a $(p-3)$-regular graph and every minimal dominating set of G is independent, then $MD(G) \cong CMD(G)$.*

Proposition 2.6 *For any signed graph $S = (G, \sigma)$, $MD(S) \sim CMD(S)$ if, and only if, G is a $(p-3)$-regular graph and every minimal dominating set of G is independent.*

Proof Suppose $MD(S) \sim CMD(S)$. This implies, $MD(G) \cong CMD(G)$ and hence by Proposition 2.5, we see that the graph G must be $(p-3)$-regular graph and every minimal dominating set of G is independent.

Conversely, suppose that G is $(p-3)$-regular graph and every minimal dominating set of G is independent. Then $MD(G) \cong CMD(G)$ by Proposition 2.5. Now, if S is a signed graph with underlying graph as $(p-3)$-regular graph and every minimal dominating set of G is independent, by Propositions 2.1 and 2.4, $MD(S)$ and $CMD(S)$ are balanced and hence, the result follows from Proposition 1.2. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [7] as follows:

$\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

Proposition 2.6 provides easy solutions to other signed graph switching equivalence relations, which are given in the following result.

Corollary 2.7 *For any signed graph $S = (G, \sigma)$, $MD(\eta(S)) \sim CMD(S)$ (or $MD(S) \sim CMD(\eta(S))$ or $MD(\eta(S)) \sim CMD(\eta(S))$) if, and only if, G is a $(p-3)$-regular graph and every minimal dominating set of G is independent.*

For a signed graph $S = (G, \sigma)$, the $MD(S)$ is balanced (Proposition 2.1). We now examine, the conditions under which negation of $MD(S)$ is balanced.

Proposition 2.8 *Let $S = (G, \sigma)$ be a signed graph. If $MD(G)$ is bipartite then $\eta(MD(S))$ is balanced.*

Proof Since, by Proposition 2.1, $MD(S)$ is balanced, each cycle C in $MD(S)$ contains even number of negative edges. Also, since $MD(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $MD(S)$ is also even. Hence $\eta(MD(S))$ is balanced. □

§3. Characterization of Minimal Dominating Signed Graphs

The following result characterize signed graphs which are minimal dominating signed graphs.

Proposition 3.1 *A signed graph $S = (G, \sigma)$ is a minimal dominating signed graph if, and only if, S is balanced signed graph and its underlying graph G is a $MD(G)$.*

Proof Suppose that S is balanced and its underlying graph G is a minimal dominating graph. Then there exists a graph H such that $MD(H) \cong G$. Since S is balanced, by Proposition 1.1, there exists a marking μ of G such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $MD(S') \cong S$. Hence S is a common dominating signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a minimal dominating signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $MD(S') \cong S$. Hence by Proposition 2.1, S is balanced. □

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