60873

Reg. No. [ ]

III Semester M.Sc. Degree Examination, April/May - 2022

MATHEMATICS

Functional Analysis

Paper : M-303 T

(CBCS Y2K17 Scheme)

(Rep)

Time : 3 Hours **Maximum Marks : 70**

Instructions to Candidates:

i. *Answer any FIVE full questions.*

ii. *All questions carry Equal marks.*

1. a. Define a normed linear space. Show that in a normed linear space following hold.

   i. Norm is continuous.

   ii. Addition and scalar multiplication is jointly continuous.

   b. If M is a closed linear subspace of a normed linear space N, then prove that N/m is a normed linear space. Further, If N is a Banach space, then show that N/m is also Banach space. **(6+8)**

2. a. Prove that the linear space $B(N, N')$ of all continuous linear transformations of N into N' is a Normed linear space, where $\|T\| = Sup\{\|Tx\| : \|x\| \leq 1\}$, $\forall T \in B(N, N')$ further show that $B(N, N')$ is complete where N' is complete.

   b. Show that there is an isometric isomorphism of a normed linear space into its second dual. **(8+6)**

3. a. If N is a normed linear space and $x_0 \in N$ with $x_0 \neq 0$, then show that there exists $f_0 \in N^*$ such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.

   b. State the open mapping theorem prove the closed graph theorem. **(6+8)**

4. a. State and prove uniform boundedness theorem.

   b. Show that the mapping $T \rightarrow T^*$ is an isometric isomorphism of B(N) into B(N*) which reverse the product and preserves the identity transformation, where T is an operator on N and T* an operator on N*. **(7+7)**

[P.T.O.

5. a. Define a Hilbert space and show that every inner product space is a normed linear space.

b. Show that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. **(7+7)**

6. a. Define orthonormal complement of a set S in a Hilbert space H. Prove the following:

i. $S \cap S^{\perp} \leq \{0\}$.

ii. $S_1 \leq S_2 \Rightarrow S_2^{\perp} \leq S_1^{\perp}$.

iii. $S^{\perp}$ is a closed sub space of H.

iv. $S \subseteq S^{\perp\perp}$.

b. If $\{e_1, e_2, ...e_n\}$ is a finite orthonormal set in a Hilbert space H and $x \in H$, then prove that

i. $\sum_{i=1}^{n} |< x, e_i >|^2 \leq \|x\|^2$.

ii. $x - \sum_{i=1}^{n} < x, e_i > e_i \perp e_j \forall_j$. **(8+6)**

7. a. Define the conjugate space H* of a Hilbert space H. For every functional f on H, prove that there is a unique $y \in H$ such that $f(x) = < x, y > \forall x \in H$.

b. Define self - adjoint operators on H, prove that the set of all self - adjoint operators in B(H) form a closed real linear space of H and contain I.

c. If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other then prove that $N_1 + N_2$ and $N_1 N_2$ are normal. **(6+4+4)**

8. a. Define unitary operator on H. Prove that T is unitary iff it is an isometric isomorphism of H onto itself.

b. State and prove spectral theorem. **(6+8)**