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Reg. No. | | | | | | | | |

III Semester M.Sc. Degree Examination, April/May - 2022

MATHEMATICS

Linear Algebra

Paper : M-304 T

(CBCS Scheme Repeaters Y2K17)

Time : 3 Hours **Maximum Marks : 70**

Instructions to Candidates:

i. Answer any **five** (5) full questions.

ii. All questions carry **equal** marks.

1. a. Define an algebra. Show that the set of all homomorphisms from V onto itself forms an algebra.

b. Define a minimal polynomial. If V is a finite dimensional vector space over F, then prove that $T \in A_F(V)$ is invertible if and only if the minimal polynomial of T is non - zero.

c. Define the rank of $T \in A_F(V)$. If V is finite dimensional over F, and for $S, T \in A_F(V)$ show that

i. $r(ST) \le r(T)$

ii. $r(TS) \le r(S)$.

iii. $r(TS) = r(ST) = r(T)$, for S regular in $A_F(V)$. **(4+4+6)**

2. a. If $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_k$ in F are distinct characteristic roots of a linear transformation $T \in A_F(V)$ and if $v_1, v_2, \ldots\ldots v_k$ are characteristic vector of T belonging to $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_k$, respectively. Then prove that $v_1, v_2, \ldots\ldots v_k$ are linearly independent over F.

b. Let V be the vector space of polynomials of degree 3 or less over F. In V define T by differential operator. Compute the matrix of T in the following bases.

i. $\{1, x, x^2, x^3\}$. Let this matrix be A.

[P.T.O.]

ii. $\{x^3, x^2, x, 1\}$. Let this matrix be B.

iii. Find a matrix C such that $B = CAC^{-1}$.

c. The element $\lambda \in F$ is a characteristic root of $T \in A_F(V)$ if and only if for some non - zero vector $v \in V$ then prove that $Tv = \lambda v$. **(5+6+3)**

3. a. Let U,V, and W be finite dimensional vector space over F. Let T be a lineear transformation from U to V and S be from V to W with respect to ordered bases B_1, B_2 and B_3. If $A = [\alpha_{ij}]$, $B = [\beta_{ij}]$ and $C = [\gamma_{ij}]$ are matrices of T, S and TS respectively in the bases B_1B_2, B_2B_3 and B_1B_3 respectively. Then prove the $C = BA$.

b. Define the change of coordinate matrix. Let $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$, be two bases with $b_1 = 4c_1 + c_2$; $b_2 = -6c_1 + c_2$. Suppose $x = 3b_1 + b_2$. Then final $[x]_C$.

c. Define a linear functional and dual basis. Let V be finite dimensional over F. Prove that there exists a unique dual basis for every basis of V. **(5+4+5)**

4. a. State and prove Cayley - Hamilton theorem.

b. Define a nilpotent transformation. If $T \in A_F(V)$ is nilpotent then prove that $\alpha_0 + \alpha_1 T + ... + \alpha_m T^m$ is invertible if $\alpha_0 \neq 0$, where $\alpha_i \in F$.

c. Prove that a nilpotent linear transformation has unique set of invariants. **(5+5+4)**

5. a. Define a basic Jordan block. Prove that two linear transformations are similar if and only if they can be brought to the same Jordan canonical form.

b. Let $T \in A_F(V)$ has a minimal polynomial $p(x) = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + ... + \gamma_{r-1} x^{r-1} + x^r$ over F. Suppose that V is a module in a cyclic module relative to T. Then prove that there exists a basis of V over F such that the matrix of T in this basis of the form

$$\begin{bmatrix} 0 & 1 & 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots & & . \\ \ldots\ldots\ldots\ldots & & . \\ \ldots\ldots\ldots\ldots\ldots & & . \\ 0 & 0 & 0 \ldots\ldots\ldots \quad 1 \\ -\gamma_0, -\gamma_1, -\gamma_2 \ldots\ldots\ldots -\gamma_{r-1} \end{bmatrix}$$

(7+7)

6. a. Let V be an inner product space over F. Then for all $x, y \in V$, prove the following

   i. $|<x, y>| \geq \|x\|.\|y\|$.

   ii. $\|x+y\| \leq \|x\| + \|y\|$.

   b. Define an orthogonal and orthonormal set. Explain Gram - Schmidt method. Apply it to find an orthonormal basis from $\left\{\begin{bmatrix}1 & -4 & 0 & 1\end{bmatrix}^T, \begin{bmatrix}7 & -4 & -4 & 1\end{bmatrix}^T\right\}$.

   c. State and prove the Bessel's inequality. **(4+6+4)**

7. a. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

   b. Define the following quadratic forms with example for each.

   i. Positive definite.

   ii. Positive semidefinite.

   iii. Negative definite.

   iv. Negative semidefinite.

   Find the nature of the quadratic form

$$QF(x) = -3x_2^2 + 4x_1^2 - 11x_1x_4 + 5x_2x_4 + 18x_1x_2 + 16x_4^2.$$

   c. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find the maximum value of quadratic form subject to $x^T.x = 1$ and find the unit vector at which this value is attained. **(6+4+4)**

8. a. Define rank and signature of a real quadratic form. Show that two real symmetric matrices are congruent if and only if they have the same rank and signature.

   b. Define bilinear and symmetric bilinear forms with example each. Let B be an bilinear form on a finite dimensional vector space V and let B be an ordered basis for V. Then show that B is symmetric of and only if $\psi_B(B)$ is symmetric. **(7+7)**