60883

Reg. No.

IV Semester M.Sc. Degree Examination, September/October - 2022

MATHEMATICS

Riemannian Geometry

(CBCS Scheme Y2K17)

Paper : M-403 T(A)

Time : 3 Hours **Maximum Marks : 70**

Instructions to Candidates:

1. Answer any **five** questions.
2. All questions carry **equal** marks.

1. a. Obtain an R^2 atlas of class C^∞ on S^2.

   b. Prove that a C^∞ atlas on a set M induces a unique topology with respect to which every chart of the atlas is a homomorphism. (8+6)

2. a. Show that the set of all real r×s matrices is a C^∞ manifold of dimension rs.

   b. Show that figure 8 has two atlasses which are not equivalent.

   c. Show that every open subset of a C^∞ manifold of dimensions n is a C^∞ manifold of dimensions n. (4+7+3)

3. a. Prove that the induced topology on a C^∞ manifold is a first axiom space and also a T_1 space.

   b. Define Lie bracket of two vector fields. Prove that Lie bracket of two vector fields is a vector field.

   c. State and prove the Jacobi identify. (6+4+4)

4. a. Define a pullback function. If $F: M_1 \to M_2$ and $G: M_2 \to M_3$ be differential maps, then prove $(G.F)^* = F^*.G^*$.

   b. Show that a derivative map is a vector space homomorphism.

   c. Prove that the tensor product of two tensors is associative, distributive over addition but is not commutative. (4+4+6)

[P.T.O.

5. a. Prove that fundamental theorem of Riemannian geometry.

b. Prove that

i. $\partial_k g^{ij} = \Gamma^{i}_{hk} g^{hj} - \Gamma^{j}_{hk} g^{hi}$.

ii. $\Gamma^{i}_{ji} = \partial_j \left(\log \sqrt{g}\right), g = \left|g_{ij}\right|$. **(8+6)**

6. a. Prove that Christoffel symbols are not components of a tensor in general.

b. Show that curvature tensor R satisfies the following properties.

i. $R(X,Y,Z,W) = -R(Y,X,Z,W)$ and hence prove $R(X,X,Z,W) = 0$.

ii. $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$.

iii. $R(X,Y,Z,W) = R(Z,W,X,Y)$.

iv. $\nabla_X R(Y,Z,U,V) + \nabla_Y R(Z,X,U,V) + \nabla_Z R(X,Y,U,V) = 0$.

c. State and prove Schur's theorem. **(3+4+7)**

7. a. Prove that a curve σ is a geodesic in M if and only if

i. tangent vector field of σ has constant length.

ii. geodesic curvature of σ is zero, i.e. $K_g = 0$.

b. Derive Gauss Codazzi equations for hypersurfaces of Riemannian manifold. **(6+8)**

8. a. Let M be a hypersurface of a Riemannian manifold and γ be a two dimensional subspaces of $T_pM, p \in M$. Let $k(\gamma)$ be the sectional curvature of M and $\overline{k(\gamma)}$ be the sectional curvature of $\overline{M}$. Let (X,Y) be orthonormal basis for γ. Then prove that $\overline{k}(\gamma) = k(\gamma) - g(LX,X)g(LY,Y) + (g(LX,Y))^2$.

b. Prove that the Weingarten map is self adjoint.

c. State and prove Gauss theorem egregium. **(5+4+5)**